HOLOMORPHIC FUNCTIONAL CALCULUS AND SOME APPLICATIONS

GONG CHEN

Abstract. In this note, we present some applications of complex analysis to functional analysis and operator theory. We will focus on some basic facts of holomorphic functional calculus and its applications to analyze the spectrum of operators.

1. Introduction

Many results from classical complex analysis like Cauchy’s theorem, Goursat’s refinement of it, Cauchy’s integral formula, Liouville theorem and etc. are powerful tools in analysis. See [WS] for detailed discussion. In this note, we will discuss some analogues of standard complex analysis results and techniques in operator theory. These results will be powerful tools for us to study operator theory and spectral theory. The rough plan of this note is as the following: In section 2, we will introduce holomorphic functional calculus which gives us many similar results as standard complex analysis, and we will see some simple applications. In section 3, we will prove the analytic Fredholm theorem and the meromorphic Fredholm theorem. They are of great importance in the study of spectral theory and scattering theory. Our standard Fredholm alternative theorem [HB] will be a corollary of the analytic Fredholm theorem when the compact operator is approximatable by finite rank operators. This holds naturally in Hilbert spaces, one can see more details in [HB]. And then in section 4, we will study Riesz projection. Riesz projection is useful when we study the isolated parts of the spectrum of an operator. We will see some application too in section 4. In the final section 5, we will apply all we get from previous sections to study a lemma in which is essential to analyze resolvent expansions at thresholds and dispersive estimates, see [JN] and [ES].

2. Holomorphic Functional Calculus

In this section, we introduce some results about vector-valued analytic functions. All these results are quite similar to standard complex analysis with complex-valued functions. It might be interesting to point out that most of the following results also hold for vector-valued functions with values in a quasi-complete, locally convex topological vector space. To pursue the general case, we need to study power series with coefficients in topological vector spaces, Gelfand-Pettis integrals and weak-to-strong principle. For simplicity, we will only consider complex Banach spaces or Hilbert spaces in this note. The discussion in this section will illustrate the main ideas for general cases.

Date: 06/09/14.
We will follow the notations in [RS1]. Let $X$ be a complex Banach space, and let $D$ be a domain in the complex plane $\mathbb{C}$. We recall that a domain $D$ is a connected open subset of $\mathbb{C}$. We want to define a notion holomorphicity for a function $x(z)$ defined on $D$ with values in $X$. Intuitively, we have two ideas in mind, first, we might think $x(z)$ is "holomorphic" at $z_0 \in D$ if the limit
\[
\lim_{h \to 0} \frac{x(z_0 + h) - x(z_0)}{h}
\]
exists in $X$ (here $h$ goes to 0 in $\mathbb{C}$) just like what we have experienced in classical complex analysis [WS]. If the above situation holds, we call $x(z)$ is strongly analytic at $z_0$. Another idea, we immediately have in mind is to define a notion of "holomorphicity" as the following: Note that $\forall \ell \in X^*$, $\ell(x(z))$ is a complex-value function, we might say $x(z)$ is holomorphic at $z_0 \in D$ if and only if $\forall \ell \in X^*$, $\ell(x(z))$ is holomorphic at $z_0$ in the sense of classical complex analysis. We say $x(z)$ is weakly analytic at $z_0$. We say $x(z)$ is strongly (weakly) analytic on $D$ if $x(z)$ is strongly (weakly) analytic at every point in $D$. At first glance, it is clear the first definition is stronger than the second definition a priori. But in fact we will show these two notations are the same, so we get a consistent notation of holomorphicity.

Remark 1. We follow the literature to make holomorphicity and analyticity interchangeable. After we make sense of complex differentiability, to identify holomorphy and analyticity is parallel to classical complex analysis. We will mention this point briefly later on.

To show the equivalence of strong and weak analyticity, we begin with a lemma. We will follow [RS1].

Lemma 2. Let $X$ be a Banach space. Then a sequence $\{x_n\}$ is Cauchy if and only if $\{\ell(x_n)\}$ is Cauchy uniformly for $\ell \in X^*$ with $\|\ell\| \leq 1$.

Proof. For one direction, we assume $\{x_n\}$ is Cauchy, then
\[
|\ell(x_n) - \ell(x_m)| \leq \|x_n - x_m\|
\]
uniformly for $\ell \in X^*$ with $\|\ell\| \leq 1$. So $\{\ell(x_n)\}$ is Cauchy uniformly for $\ell \in X^*$ with $\|\ell\| \leq 1$. For the other direction, with the help of Hahn-Banach theorem [HB],
\[
\|x_n - x_m\| = \sup_{\|\ell\| \leq 1} |\ell(x_n) - \ell(x_m)|.
\]
So, if $\{\ell(x_n)\}$ is Cauchy uniformly for $\ell \in X^*$ with $\|\ell\| \leq 1$, then $\{x_n\}$ is Cauchy in norm. \hfill \square

As mentioned above, clearly strong analyticity implies weak analyticity. Now we show the other direction.

Theorem 3. Every weakly analytic function is strongly analytic.

Proof. Let $x(z)$ be a weakly analytic function on $D$ with values in $X$ as above. Let $z_0$ be an arbitrary point in $D$ and let $\Gamma$ be a circle surrounding $z_0$ and the interior of it is contained in $D$. By our assumption, $\forall \ell \in X^*$, $\ell(x(z))$ is analytic. Now by
the standard Cauchy formula from classical complex analysis [WS], we have
\[
\ell \left( \frac{x(z_0 + h) - x(z_0)}{h} \right) - \frac{d}{dz} \ell(x(z_0)) = \frac{1}{2\pi i} \oint_{\Gamma} \left[ \frac{1}{h} \left( \frac{1}{z - (z_0 + h)} - \frac{1}{z - z_0} \right) - \frac{1}{(z - z_0)^2} \right] \ell(x(z)) \, dz.
\]

Now since \( \ell(x(z)) \) is continuous on \( \Gamma \) and \( \Gamma \) is compact, \( \| \ell(x(z)) \| \leq C_{\ell} \) for all \( z \in \Gamma \) for some constant \( C_{\ell} \). Now we observe that \( x(z) \) is a family of maps from \( X^* \) to \( \mathbb{C} \) by \( \ell(x(z)) \) and \( x(z) \) is pointwise bounded at each \( \ell \), so by the uniform boundedness principle we have \( \sup_{z \in \Gamma} \| x(z) \| \leq C < \infty \) for some constant \( C \). Thus
\[
\left| \ell \left( \frac{x(z_0 + h) - x(z_0)}{h} \right) - \frac{d}{dz} \ell(x(z_0)) \right| \leq \frac{\| \ell \|}{2\pi} \left( \sup_{z \in \Gamma} \| x(z) \| \right) \frac{1}{h} \left( \frac{1}{(z_0 + h) - z} - \frac{1}{z - z_0} \right) \, dz.
\]

This implies \( \ell \left( \frac{x(z_0 + h) - x(z_0)}{h} \right) \) is uniformly Cauchy for \( \| \ell \| \leq 1 \) (More precisely, we should apply uniform Cauchy for some specific sequence \( \{ h_n \} \) goes to 0, then apply the triangle inequality to get the general case). By Lemma 2, \( \frac{x(z_0 + h) - x(z_0)}{h} \) converges in \( X \), \( x(z) \) is strongly analytic at \( z_0 \). Since \( z_0 \) is arbitrary, so \( x(z) \) is strongly analytic on \( D \). \( \square \)

Now, we have a consistent way to define holomorphic functions with vector-values (operator-values). Here, let us see some theorems parallel to classical complex analysis. We will see the power of weak analyticity in the proofs.

Let \( x(z) \) a function analytic in the domain \( D \subset \mathbb{C} \) and \( \Gamma \) be a finite union of regular piecewise smooth curves contained in \( D \). We can define the integral \( \oint_{\Gamma} x(z) \, dz \) along the curve \( \Gamma \) defined in the standard way, say, via the regulated integral. As in classical complex analysis, it is called the Cauchy integral (actually it is enough to request \( \Gamma \) to be a rectifiable curve). We have the Cauchy theorem just like in the classical case.

**Theorem 4.** (Cauchy). Let \( \Gamma = \partial D \) with \( D \) simply connected. Suppose \( x(z) \) is analytic in a neighborhood of \( D \). Then
\[
\oint_{\Gamma} x(z) \, dz = 0.
\]

**Proof.** Suppose \( \oint_{\Gamma} x(z) \, dz = y \in X \). For arbitrary \( \ell \in X^* \), we have \( \ell \left( \oint_{\Gamma} x(z) \, dz \right) = \ell(y) = \oint_{\Gamma} \ell(x(z)) \, dz \) by the definition of the integral. Then by the analyticity of \( x \), we know \( \ell(x(z)) \) is an analytic function in the classical sense. So \( \ell \left( \oint_{\Gamma} x(z) \, dz \right) = \ell(y) = \oint_{\Gamma} \ell(x(z)) \, dz = 0 \). Since \( \ell \) is arbitrary, we can conclude \( y = 0 \). \( \square \)

Following the same argument in classical complex analysis, we can also get the integral representation of \( x(z) \).

**Theorem 5.** Under the same conditions as Theorem 4, we have
\[
x(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{x(\xi)}{\xi - z} \, d\xi,
\]
\[x^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{x(\xi)}{(\xi - z)^{n+1}} d\xi,\]

where \(n\) is a non-negative integer.

With the help of Theorem 5, we can show in the domain of holomorphicity of \(x(z)\), we can have a power series representation for \(x(z)\) at least in some small neighborhood of each point of \(D\). All the proofs are just identical to classical sense. So we can identify holomorphy and analyticity just like our remark 1.

**Theorem 6.** (Liouville) Let \(x(z)\) be an analytic function for all \(z \in \mathbb{C}\) (entire functions). If \(x(z)\) is uniformly bounded, i.e, \(\exists C > 0\) so that \(\|x(z)\| < C\) for all \(z \in \mathbb{C}\), then \(x(z)\) is a constant.

**Proof.** \(\forall \ell \in X^\ast\), from the standard Liouville’s theorem [WS], we know \(\ell(x(z))\) is a constant. So \(\ell(x(z)) = C_l\) for some constant \(C_l\). Now we fix one point \(z_0 \in \mathbb{C}\), \(\ell(x(z_0)) = \ell(x(z_0))\) for arbitrary \(\ell \in X^\ast\). So we can conclude \(x(z)\) is a constant. \(\square\)

Next, we apply the above results to spectral theory. First of all, let us define the spectrum.

**Definition 7.** The set \(\rho(x)\) is the collection of all points \(z \in \mathbb{C}\) so that \(x - z\) invertible. The set \(\rho(x)\) is called the resolvent set of \(x\). The set \(\sigma(x)\) of all \(z \in \mathbb{C}\) such that \(x - z\) is not invertible is called the spectrum of \(x\). In particular, when our Banach space is the space of linear operators, we have the following definitions:

Let \(A\) be a linear operator from \(X\) to \(X\). The resolvent set \(\rho(A)\) consists of \(z \in \mathbb{C}\) such that \(A - z: D(A) \to X\) is one-to-one and its inverse, \((A - z)^{-1}\), is bounded. In particular, \(D((A - z)^{-1}) = X\), \(\text{Range}((A - z)^{-1}) \subset D(A)\). The spectrum of an operator is defined by

\[\sigma(A) = \mathbb{C} \setminus \rho(A).\]

In particular we have the following two specific classes.

Discrete spectrum: \(\sigma_d(A)\) consists of isolated points of the spectrum which correspond to eigenvalues with finite algebraic multiplicity.

Essential spectrum: \(\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A)\).

A beautiful application of Theorem 6 is the following:

**Theorem 8.** The spectrum of every \(x \in X\) is not empty.

**Proof.** Suppose for some \(x \in X\), \(\sigma(x) = \emptyset\). Then \(\frac{1}{x-z}\) is analytic for all \(z \in \mathbb{C}\). For all \(z \in \mathbb{C} \setminus \{0\}\), we have

\[\left\| \frac{1}{x-z} \right\| = \left\| \frac{1}{z} \right\| \left\| (I - \frac{x}{z})^{-1} \right\|.\]

Note that when \(z\) goes to \(\infty\),

\[(I - \frac{x}{z})^{-1} \to I^{-1} = I.\]

So we can see \(\left\| \frac{1}{x-z} \right\|\) is bounded and by Theorem 6, \(\frac{1}{x-z}\) is a constant. Let \(z\) goes to \(\infty\), we get \(\frac{1}{x-z} = 0\). It is a contradiction. \(\square\)
3. Fredholm Alternative

In this section, we will approach the Fredholm theorems by holomorphic functional calculus. The classical Fredholm alternative will be a corollary of the analytic Fredholm theorem in this section in case the compact operator can be approximated by finite rank operators. The basic philosophy of the Fredholm alternative is basically that compactness and uniqueness imply existence. In finite dimension case, this is trivial for all linear operator based on rank-nullity theorem. But in general case, e.g. for Hilbert and Banach spaces, it is not trivial and in general without the compactness assumption it false [HB]. In this section, we will follow [RS1] and [RS4]. The materials covered by [RS1] are based on Hilbert space structure, we will see we can use a trick to get the results for the situation when the compact operator can be approximated by finite rank operators in a Banach space.

First of all let $X$ be a Hilbert space. It is well-known any compact operator in $\mathcal{L}(X)$ can be approximated by finite rank operators. Then we have the analytic Fredholm theorem.

**Theorem 9.** (Analytic Fredholm theorem) Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathcal{L}(X)$ be an analytic operator-valued function, with $f(z)$ compact for each $z \in \Omega$. Then either

(a) $(I - f(z))^{-1}$ exists for no $z \in D$, \\

or

(b) $(I - f(z))^{-1}$ exists for all $z \in D\setminus S$ where $S$ is a discrete subset of $D$. In this case, $(I - f(z))^{-1}$ is analytic in $D \setminus S$ and meromorphic in $D$ with finite rank operators at negative powers of $z - z_0$ (analogue of Laurent series) for $z_0 \in S$. In particular, the residues at the poles are finite rank operators. At $z \in S$, $\ker(I - f(z)) \neq \{0\}$.

**Proof.** It suffices to show near any $z_0 \in D$, either (a) or (b) holds. Then, applying a standard connectedness argument, we can convert this into a statement about all of $D$. Given $z_0 \in D$, chose $r$ small so that when $|z - z_0| < r$, $\|f(z_0) - f(z)\| < \frac{1}{2}$. Let $F$ be a finite rank operator such that

$$\|f(z_0) - F\| < 1/2.$$ 

For $z \in D_r$, a disc of radius $r$ centered at $z_0$, $\|f(z) - F\| < 1$. We can see $(1 - f(z) + F)^{-1}$ exists and is analytic by expanding it as a geometric series as in section 2. Note that

$$1 - f(z) = (1 - F(1 - f(z) + F)^{-1})(1 - f(z) + F).$$

Define $g(z) = F(1 - f(z) + F)^{-1}$. Notice that $1 - f(z)$ is invertible if and only if $1 - g(z)$ is invertible and that $\psi = f(z)\psi$ has a nonzero solution if and only if $\varphi = g(z)\varphi$ has a nonzero solution.

Since $F$ is of finite rank, there are independent vectors $\psi_1, \ldots, \psi_n$ so that $F(\varphi) = \sum_{n=1}^{N} \alpha_n(\varphi)\psi_n$, where the $\alpha_n$’s are bounded linear functionals on $X$. So by the Riesz presentation theorem, there are vectors $\phi_1, \ldots, \phi_n$, such that $F(\varphi) = \sum_{n=1}^{N} (\phi_i, \varphi)\psi_i$ for all $\varphi \in X$. Let $\phi_n(z) = \left((1 - f(z) + F)^{-1}\right)^*$. Then we can write

$$g(z) = \sum_{n=1}^{N} (\phi_n(z), \varphi)\psi_n.$$
Now if \( g(z)\varphi = \varphi \), then if \( \varphi = \sum_{n=1}^{N} \beta_n \psi_n \), we can find \( \beta \) that satisfies

\[
(3.1) \quad \beta_n = \sum_{m=1}^{N} (\phi_n(z), \psi_m) \beta_m.
\]

On the other hand, if the equation 3.1 has a solution \( \langle \beta_1, \ldots, \beta_N \rangle \) then \( \varphi = \sum_{n=1}^{N} \beta_n \psi_n \), is a solution of \( g(z)\varphi = \varphi \). Thus, \( g(z)\varphi = \varphi \) has a non-trivial solution if and only if the determinant

\[
(3.2) \quad d(z) = \det \{ \delta_{nm} - (\phi_n(z), \psi_m) \} = 0.
\]

We can see \( (\phi_n(z), \psi_m) \) is analytic in \( D_r \) so \( d(z) \) is also analytic. So it means that either \( S_r = \{ z \mid z \in D_r, d(z) = 0 \} \) is a discrete set in \( D_r \) or \( S_r = D_r \). Now, if we suppose \( d(z) \neq 0 \), then, given \( \psi \), we can solve \( (1 - g(z))\varphi = \psi \) by setting \( \varphi = \psi + \sum_{n=1}^{N} \beta_n \psi_n \) if we can solve

\[
(3.3) \quad \beta_n = (\phi_n(z), \psi) + \sum_{m=1}^{N} (\phi_n(z), \psi_m) \beta_m.
\]

Now, we have already assumed \( d(z) \neq 0 \), so the above equation has a solution. So \( (1 - g(z))^{-1} \) exists if and only if for \( z \notin S_r \).

Notice that \( 1 - g(z) \) restricted to \( \text{Range}(F) \) is a finite rank operator, so \( d(z) \) can not have zero of order more than the dimension of \( \text{Range}(F) \). The meromorphic nature of \( (1 - f(z))^{-1} \) and the finite rank residues follow from the fact there is an explicit formula for \( \beta_n \) in equation 3.3 in terms of cofactor matrices. \quad \Box

**Remark 10.** In the above theorem and proof, the coefficients of negative power of \( z - z_0 \) means the coefficients of the Laurent series of the \( (1 - f(z))^{-1} \) near poles. That those coefficients are of finite rank can be seen just by the fact \( 1 - g(z) \) restricted to \( \text{Range}(F) \) is just a finite dimension operator and the standard construction.

**Remark 11.** In the proof above, we used the fact \( X \) is a Hilbert space so that we can apply Riesz representation. We can use an analytic (meromorphic) continuation argument to show similar argument as above holds for more general Banach spaces. It suffices to consider the finite rank case, the general case is just an approximation argument similarly as above. Suppose \( A \) is a finite rank operator, then

\[
R_A(z) = (z - A)^{-1} = \frac{1}{z} \left( 1 + \frac{1}{z} A + \frac{1}{z^2} A + \ldots \right) = \frac{1}{z} I + \frac{1}{z} R_A(z),
\]

for \( |z| > \|A\| \). Let \( R_N = (zI_N - A_N)^{-1} \) be the finite dimensional resolvent of \( A_N \) where \( A_N \) is \( A \) restricted to its range. Then we can verify \( R_N \) is a meromorphic function with finite order poles at the spectrum of \( A_N \). We can define

\[
S(z)\phi = \frac{1}{z} \phi + \frac{1}{z} R_N(z)\phi.
\]

Finally, it is easy to check \( S(z) \) is a meromorphic continuation of \( R_A(z) \). The claims in the proof are reduced to \( S \).

**Remark 12.** It is not surprising that the similar results hold for an operator \( A \) such that \( A^k \) is compact for some \( k \) is a positive integer. One can find the details in [DRY].

Now the classical Fredholm alternative is just a corollary of our Theorem 9.
Corollary 13. (Fredholm alternative) If $A$ is a compact operator on $X$, then either $(I - A)^{-1}$ exists or $A\psi = \psi$ has a solution.

Proof. Take $f(z) = zA$ and apply our Theorem 9 at $z = 1$. □

The standard results of the W Riesz-Schauder theorem and the Hilbert-Schmidt theorem also just follow from our corollary easily. One can find more details in [HB] and [RS1]. Another nice application of the above theorem is Weyl’s theorem on locating the essential spectrum for the simple case: absence of discrete spectrum. One can find a detailed proof in [RS4].

Theorem 14. (Weyl’s theorem on essential spectrum [RS4]) Let $A$ and $B$ be bounded self-adjoint operators, with $\sigma_d(A) = \sigma_d(B) = \emptyset$ and with $A - B$ compact. Then

$$\sigma_{ess}(A) = \sigma_{ess}(B).$$

Now we present the more general version of the Fredholm alternative following [RS4]. Again $X$ is a Hilbert space.

Theorem 15. (Meromorphic Fredholm theorem) Let $\Omega \subset \mathbb{C}$ be open and connected, and let $A : \Omega \to \mathcal{L}(X)$ be a meromorphic operator-valued function of $z$, i.e., $A$ is analytic in $\Omega \setminus D$ where $D$ is a discrete subset of $\Omega$ and near any $z_0 \in D$,

$$A(z) = A_{-k}(z - z_0)^{-k} + \ldots + A_{-1}(z - z_0)^{-1} + \sum_{n=0}^{\infty} A_n(z - z_0)^n.$$

Suppose in addition that:

1. $A(z)$ is compact if $z \in \Omega \setminus D$.
2. The coefficients $A_{-k}, \ldots, A_{-1}$ of the negative terms of the Laurent series of $A(z)$ at points $z_0 \in D$ are finite rank operators.

Then either:

(a) $1 - A(z)$ is invertible for no $z \in \Omega \setminus D$.

or

(b) There is a discrete set $D' \subset \Omega$ so that $(I - A(z))^{-1}$ exists for $z \notin D \cup D'$ and extends to a function analytic in $\Omega \setminus D'$ and meromorphic in $\Omega$ such that the coefficients of the negative terms in the Laurent series at the point $z_0 \in D'$ are finite rank operators.

Proof. The proof of the theorem is closely related to the proof of Theorem 9 above. The only difference is now we use finite rank operators to approximate $G(z)$ where $G(z)$ is compact and is defined as $A(z) = A_{-k}(z - z_0)^{-k} + \ldots + A_{-1}(z - z_0)^{-1} + G(z)$ for $z \neq z_0$. By the continuity of $G$, we can also define $G(z_0)$ which is also compact as a limit of compact operators. We omit the details of the proof here, the details are more or less similar to the proof above. One can find detailed proof in [RS4] □

Finally, we present a theorem based on our Theorem 15. It is a more general case of Weyl’s theorem.

Theorem 16. (Weyl’s theorem on essential spectrum; [RS4]) Let $A$ be self-adjoint and $B$ be bounded, such that for some (and hence for all) $z \in \rho(A) \cap \rho(B)$ the operator $(A - z)^{-1} - (B - z)^{-1}$ is compact and either

$\sigma(A) \neq \mathbb{R}$, $\rho(B) \neq \emptyset$

or

There are points of $\rho(B)$ in both the upper and lower half-plane.
then
\[ \sigma_{ess}(A) = \sigma_{ess}(B). \]

4. Riesz Projections

In this section, we will study a very important tool, the Riesz projections to analyze the isolated part the spectrum of an operator. To motivate the construction, let us first consider the finite dimensional case.

Suppose \( \mathcal{H} \) is a finite dimensional vector space. Let \( A \) be a linear operator from \( \mathcal{H} \) to itself, i.e., \( A \) is a matrix. Let \( \lambda_1, \ldots, \lambda_r \) be the distinct eigenvalues of \( A \). Then we can find a basis to write \( A \) as its Jordan canonical form \( J_A \). So \( J_A \) is a block diagonal matrix, and each of these blocks on the diagonal are of the form

\[
T_j = \begin{bmatrix}
\lambda_j & 1 \\
 & \ddots & \ddots \\
& & \ddots & 1 \\
& & & \lambda_j
\end{bmatrix}.
\]

We define \( M_j \) to be the subspace spanned by the basis vectors associated to the block \( T_j \) with eigenvalue \( \lambda_j \) on the main diagonal. Then we know we can decompose \( \mathcal{H} \) as:

\[
\mathcal{H} = M_1 \oplus \ldots \oplus M_r.
\]

Each \( M_i \) is invariant under \( A \) and the restriction of \( A \) to \( M_i \) has a single eigenvalue \( \lambda_i \).

In order to find \( M_i \), one explicit construction is to find the basis associated to it. This method certainly could not be easily generalized to infinitely dimensional space. Now we introduce a contour integration to find \( M_i \). In fact

\[
M_i = \text{Image} \left[ \frac{1}{2\pi i} \oint_{\Gamma_i} \! (z - A)^{-1} dz \right],
\]

where \( \Gamma_i \) is a contour around \( \lambda_i \) separating \( \lambda_i \) from all the other eigenvalues of \( A \). To confirm the construction, we note that for the \( k \times k \) matrix \( T_j \) given by 4.1, we have

\[
(z - T_j)^{-1} = \begin{bmatrix}
(z - \lambda_j)^{-1} & (z - \lambda_j)^{-2} & \cdots & (z - \lambda_j)^{-k} \\
& (z - \lambda_j)^{-1} & \ddots & \vdots \\
& & \ddots & (z - \lambda_j)^{-2} \\
& & & (z - \lambda_j)^{-1}
\end{bmatrix}.
\]

Therefore \( \frac{1}{2\pi i} \oint_{\Gamma_i} (z - T_j)^{-1} dz \) is a \( k \times k \) identity matrix if \( j = i \) and zero matrix otherwise. Hence

\[
\left\{ \frac{1}{2\pi i} \oint_{\Gamma_i} \! (z - A)^{-1} dz \right\} x = \begin{cases} 
  x & x \in M_i \\
  0 & x \notin M_i
\end{cases}.
\]

We can see in this construction, the finite dimensionality is not crucial. So we can generalize this process to the infinite dimension situation.
In this section, we use $X$ as a Banach space, and $A$ is a linear operator from $X$ to itself. We want to develop an analogue decomposition as above. But in the infinite dimensional case, the spectrum of $A$ is not always discrete as above. So we need to be cautious here. A set $\sigma$ is called an isolated part of $\sigma(A)$ if both $\sigma$ and $\tau := \sigma(A) \setminus \sigma$ are closed. We define the Riesz projector corresponding to $\sigma$ by

$$P_\sigma = \frac{1}{2\pi i} \oint_\Sigma \frac{dz}{z - A},$$

where $\Sigma$ is a Cauchy contour around $\sigma$ separating $\tau$. Clearly, $P_\sigma$ is a bounded linear operator. The proof of the existence of such contour can be found in the last section of Chapter 2 in [WS]. First, we notice the definition is independent of the choice of $\Gamma$ by a standard complex analysis argument. Notice that we can regard $P_\sigma$ as the residue of $(z - A)^{-1}$ at some pole if $\sigma$ is an isolated point.

**Lemma 17.** $P_\sigma$ does not depend on the contour $\Gamma$ (as long as the contour satisfies the above requirements).

We call $P_\sigma$ the Riesz projection of $A$ associated to the isolated part $\sigma$. Now we justify $P_\sigma$ is a projection.

**Lemma 18.** The operator $P_\sigma$ is a projection, i.e., $P_\sigma^2 = P_\sigma$.

**Proof.** Let $\Gamma_1$ and $\Gamma_2$ be two contours around $\sigma$ separating $\sigma$ from $\tau$. Without loss of generality, we assume $\Gamma_1$ is in the interior of the bounded domain bounded by $\Gamma_2$. Then

$$P_\sigma^2 = \left( \frac{1}{2\pi i} \oint_{\Gamma_1} \frac{dz}{z - A} \right) \left( \frac{1}{2\pi i} \oint_{\Gamma_2} \frac{dz}{z - A} \right) = \left( \frac{1}{2\pi i} \right)^2 \int_{\Gamma_1} \int_{\Gamma_2} \frac{d\lambda}{\lambda - A} \frac{dz}{z - A}.$$

We apply the resolvent equation,

$$(\lambda - A)^{-1} - (z - A)^{-1} = (z - \lambda)(\lambda - A)^{-1}(z - A)^{-1},$$

for $\lambda, z \in \rho(A)$. So we can write $P_\sigma^2 = A - B$ where

$$A = \left( \frac{1}{2\pi i} \right)^2 \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{z - \lambda} (\lambda - A)^{-1} \frac{dz}{dz} d\lambda = \frac{1}{2\pi i} \int_{\Gamma_1} (\lambda - A)^{-1} \left( \frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{z - \lambda} \frac{dz}{dz} \right) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_1} (\lambda - A)^{-1} = P_\sigma.$$
To analyze $B$,

\[
B = \left( \frac{1}{2\pi i} \right)^2 \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{z - \lambda} (z - A)^{-1} dz d\lambda \\
= \left( \frac{1}{2\pi i} \right)^2 \int_{\Gamma_2} \int_{\Gamma_1} \frac{1}{z - \lambda} (z - A)^{-1} d\lambda dz \\
= \frac{1}{2\pi i} \int_{\Gamma_2} (z - A)^{-1} \left( \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{z - \lambda} d\lambda \right) dz = 0.
\]

We used $\int_{\Gamma_2} \frac{dz}{z - \lambda} = 2\pi i$ for $\lambda \in \Gamma_1$, and $\int_{\Gamma_1} \frac{d\lambda}{z - \lambda} = 0$ for $z \in \Gamma_2$. Also we interchanged the integrals in the computation which just a result of Fubini’s theorem since the integrand is continuous on $\Gamma_1 \times \Gamma_2$. \qed

Now we can carry out a decomposition like what we have in finite dimensional case.

**Theorem 19.** Let $\sigma$ be an isolated part of $\sigma(A)$, let $M = \text{Image} P_\sigma$ and $L = \text{Ker} P_\sigma$. Then $X = M \oplus L$, and $M$ and $L$ are $A$ invariant subspaces. Also we have

\[(4.4) \quad \sigma(A|M) = \sigma, \quad \sigma(A|L) = \sigma(A) \setminus \sigma.\]

**Proof.** From the above lemma, we know $P_\sigma$ is a projection, so it is clear $M$ and $L$ are closed subspaces and $X = M \oplus L$. $A(z - A)^{-1} = (z - A)^{-1}A$ for $z \in \rho(A)$, it follows that $AP_\sigma = P_\sigma A$. So we also get $M$ and $L$ are invariant under $A$. Now we prove 4.4.

Let $\Gamma$ be a Cauchy contour around $\sigma$ separating $\sigma$ from $\tau = \sigma(A) \setminus \sigma$. Basically, for $\mu \notin \sigma$, we want to construct an inverse of $(\mu - A)$ when we consider its restriction to $M$. Intuitively, we might just consider

\[S(\mu) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\mu - z} (z - A)^{-1} dz.\]

By the Cauchy formula, formally $S(\mu)$ is just $(\mu - A)^{-1}$. More precisely, we consider for $\mu \notin \Gamma$, we can define $S(\mu)$, so $P_\sigma$ commutes with $A$ and $P_\sigma$ commutes with $(z - A)^{-1}$ for $z \in \rho(A)$. So $P_\sigma$ commutes with $S(\mu)$. So $M$ and $L$ are invariant under $S(\mu)$. We do the following computation:

\[
S(\mu)(A - \mu) = (A - \mu)S(\mu) \\
= \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\mu - z} (A - \mu)(z - A)^{-1} dz \\
= \frac{1}{2\pi i} \int_{\Gamma} \frac{-1}{\mu - z} Idz - \frac{1}{2\pi i} \int_{\Gamma} (z - A)^{-1} dz \\
= \begin{cases} 
I - P_\sigma & \mu \text{ inside } \Gamma \\
-P_\sigma & \mu \text{ outside } \Gamma
\end{cases}.
\]
Take $\mu \notin \sigma$, we always take $\Gamma$ such that $\mu$ is outside of $\Gamma$ (here we mean $\mu$ is outside of the region bounded by $\Gamma$). Then the above calculation implies

$$(A - \mu)S(\mu)x = S(\mu)(A - \mu)x = -x,$$

for $x \in M$.

Now, since $S(\mu)M \subset M$, it follows that $A - \mu$ maps $M$ one-to-one and onto $M$. Also $(u - A|M)^{-1} = S(\mu)|M$. So $\mu \in \rho(A|M)$, and we can conclude that $\sigma(A|M) \subset \sigma$. Similarly, $\sigma(A|L) \subset \tau$.

Finally, take $\lambda \notin \sigma(A|M) \cup \sigma(A|L)$. Then $\lambda - A$ maps $M$ bijectively to $M$. Since $X = M \oplus L$, so we know $\lambda \in \rho(A)$. Therefore

$$\sigma(A) \subset \sigma(A|M) \cup \sigma(A|L) \subset \sigma \cup \tau = \sigma(A).$$

Hence we get

$$\sigma(A|M) = \sigma, \quad \sigma(A|L) = \sigma(A) \setminus \sigma.$$

\[ \square \]

**Corollary 20.** Assume $\sigma(A)$ is a disjoint union of two closed subsets $\sigma$ and $\tau$, then $P_\sigma + P_\tau = I$ and $P_\sigma \cdot P_\tau = 0$.

It is clear the above corollary is the generalized situation of the finite dimension case.

Finally, we discuss some specific cases when $\sigma$ is an isolated point and $X$ is a Hilbert space. We assume $\sigma = \lambda_0$.

**Lemma 21.** Assume $X$ is a Hilbert space, and let $\sigma$ be an isolated part of $\sigma(A)$. Then $\sigma = \{ \lambda | \lambda \in \sigma \}$ is an isolated part of $\sigma(A^*)$ and

$$(P_\sigma(A))^* = P_\sigma(A^*).$$

**Proposition 22.** Range $P_{\lambda_0} \supset \ker(A - \lambda_0)$. If $A$ is self-adjoint, then $P_{\lambda_0}$ is an orthogonal projection onto $\ker(A - \lambda_0)$. In particular, Range $P_{\lambda_0} = \ker(A - \lambda_0)$.

Proof. Let us check that Range $P_{\lambda_0} \supset \ker(A - \lambda_0)$. Let $(A - \lambda_0)f = 0$. Then

$$P_{\lambda_0}f = \frac{1}{2\pi i} \oint_{\Gamma_{\lambda_0}} (z - A)^{-1} f \, dz = \frac{1}{2\pi i} \oint_{\Gamma_{\lambda_0}} (z - \lambda_0)^{-1} f \, dz = f.$$

Next, let us check that if $A$ is self-adjoint, then Range $P_{\lambda_0} \subset \ker(A - \lambda_0)$ and indeed $P_{\lambda_0}$ is an orthogonal projection. We notice that from lemma 21,

$$(P_\tau(A))^* = P_\tau(A)$$

since $A$ is self-adjoint. So in particular, $P_{\lambda_0}$ is an orthogonal projection if $A$ is self-adjoint. To show Range $P_{\lambda_0} \subset \ker(A - \lambda_0)$, we consider

$$\begin{align*}
(A - \lambda_0)P_{\lambda_0} &= \frac{1}{2\pi i} \oint_{\Gamma_{\lambda_0}} (A - \lambda_0)(A - z)^{-1} \, dz \\
&= \frac{1}{2\pi i} \oint_{\Gamma_{\lambda_0}} (z - \lambda_0)(A - z)^{-1} \, dz.
\end{align*}$$

Let $U_{\lambda_0}$ denote the interior of $\Gamma_{\lambda_0}$. On $U_{\lambda_0} \setminus \{ \lambda_0 \}$, the operator $(z - \lambda_0)(A - z)^{-1}$ is an analytic, operator-valued function and satisfies the bound
\[ |(z - \lambda_0)| \|A - z\|^{-1} \leq |z - \lambda_0| d(z, \sigma(A))^{-1}, \]
where \(d(x, y)\) is the distance from \(x\) to \(y\). We know for a self-adjoint operator \(\|A - z\|^{-1} = \text{dist}(z, \sigma(A))\). Now we take the diameter of \(\Gamma_{\lambda_0}\) small enough so that \(\lambda_0\) is the closest point of \(\sigma(A)\) to \(\Gamma_{\lambda_0}\). Consequently, \(|(z - \lambda_0)| \|A - z\|^{-1} \leq 1\) and this function is uniformly bounded on \(U_{\lambda_0} \setminus \{\lambda_0\}\). By the standard argument in classical complex analysis, we know we can extend \((z - \lambda_0)(A - z)^{-1}\) to be an analytic function on \(U_{\lambda_0}\). Finally by Cauchy’s theorem, the integral vanishes. Therefore \(\text{Range } P_{\lambda_0} = \ker(A - \lambda_0)\).

**Remark 23.** By a similar argument as above, we can show \(\text{Range } (P_{\lambda_0}) \supset \bigcup_{n=1}^{\infty} \ker((A - \lambda_0)^n)\).

We end this section by some interesting propositions of the above results without proof.

**Proposition 24.** The isolated points of \(\sigma(A)\), \(A = A^*\) are eigenvalues of \(A\).

**Proposition 25.** Let \(\lambda_0\) be a isolated point of \(\sigma(A)\) and \(A\) be self-adjoint. Then \((A - \lambda)\) restricted to \([\ker(A - \lambda)]^\perp\) has a bounded inverse.

## 5. Application

In this section, we analyze a lemma from [JN] which is essential to study resolvent expansions at thresholds and dispersive estimate. We will follow the process in [ES]. In this section \(H\) is a Hilbert space. We will first state the lemma in the form [JN] but we will proceed everything following the form in [ES].

**Lemma 26.** Let \(A\) be a closed operator and \(S\) is a projection. Suppose \(A + S\) has a bounded inverse. Then \(A\) has a bounded inverse if and only if

\[ a \equiv S - S(A + S)^{-1}S \]

has a bounded inverse in \(SH\), and in this case

\[ A^{-1} = (A + S)^{-1} + (A + S)^{-1}Sa^{-1}S(A + S)^{-1}. \]

Now we formulate the lemma in the form [ES] uses.

**Lemma 27.** Let \(F \subset \mathbb{C} \setminus \{0\}\) have \(z = 0\) as an accumulation point. Let \(A(z), z \in F\), be a family of bounded operator of the form

\[ A(z) = A_0 + zA_1(z) \]

with \(A_1(z)\) uniformly bounded as \(z \to 0\). Suppose that 0 is an isolated point of the spectrum of \(A_0\), and let \(S\) be the corresponding Riesz projection. Assume that \(\text{rank}(S)\) is finite. Then for sufficiently small \(z \in F\) the operators

\[ B(z) := \frac{1}{z}(S - S(A(z) + S)^{-1}S) \]

are well-defined and bounded on \(H\). Moreover, if \(A_0 = A_0^*\), then they are uniformly bounded as \(z \to 0\). The operator \(A(z)\) has a bounded inverse in \(H\) if and only if \(B(z)\) has a bounded inverse in \(SH\), and in this case
Before we give the proof, let us analyze some ideas behind the lemma. Basically, we have an operator $A$ and a projection $S$. A priori, we do not know if $A$ is invertible, also we know the projection is not necessarily to be one-to-one. But we know $G = A + S$ is invertible, and we want to use this information to construct the inverse of $A$. Given $G$ is invertible and an element $x$ in $\mathcal{H}$, we can look at the preimage of $Sx$ under $G$ and consider the difference of $x - G^{-1}Sx$ which measures the difference of the “preimage” of $Sx$ under $S$, i.e., $x$ and $G^{-1}Sx$. Then what $a$ tells us is the image of the difference, $x - G^{-1}Sx$, under projection $S$. So if $\text{SH}$ is nice enough, for any vector $y$ in $\text{SH}$, we can use $a$ is invertible to write $y = Sx - SG^{-1}Sx$ for some $x$ in $\mathcal{H}$. Now, we want to construct the inverse of $A$. Given $Ax$, $G$ is invertible and $a$ is invertible. The only thing we can do is apply $G^{-1}$ to $Ax$. We consider the difference between $x$ and $G^{-1}Ax$. Since $G$ is invertible, we can apply $G$ to $x - G^{-1}Ax$, we get $G(x-G^{-1}Ax) = Sx$.

To obtain $Sx$, we know $a$ is invertible, $a(Sx) = Sx - S(A + S)^{-1}Sx = (A + S)^{-1}Ax$ since $S$ is a projection. So the information of $Sx$ can be obtained from our data, $a^{-1}S(A + S)^{-1}Ax = Sx = G(x-G^{-1}Ax)$. So here we get one-side inverse of $A$. $x = G^{-1}Ax + G^{-1}a^{-1}SG^{-1}Ax$. $S$ is a projection, then in order to get a symmetric form of above construction, we note that $Sa^{-1} = a^{-1}$ since $a^{-1}$ is from $\text{SH}$ to $\text{SH}$. We apply $S$ to $a^{-1}S(A + S)^{-1}Ax = Sx = G(x - G^{-1}Ax)$, i.e., $Sx = Sa^{-1}S(A + S)^{-1}Ax$. So $x = G^{-1}Ax + G^{-1}Sa^{-1}SG^{-1}Ax$. Then we can easily check $G^{-1} + G^{-1}Sa^{-1}SG^{-1}$ is two-side inverse. Intuitively, $AG^{-1} + AG^{-1}Sa^{-1}SG^{-1} = (G - S)G^{-1} + (G - S)G^{-1}Sa^{-1}SG^{-1} = I - SG^{-1} + (I - SG^{-1})Sa^{-1}SG = I$.

Conversely, if $A$ is invertible, the way to think about above construction is similar. The observation should be there are two operators, we know one of them is invertible, the “measurement” of the difference is invertible if and only if the other operator is invertible because we can construct the inverse of each other based on the “measurement” of difference from an invertible operator.

The idea to apply above lemmas is to find appropriate sequence of spaces and find appropriate projections, then we can reduce the invertibility of an operator to an operator living a smaller scale. We can do some finite induction argument. If in one scale, the operator is invertible, then we can push it back to our original problem.

Now we start to prove the lemma, we follow the discussion in [ES].

Proof. From our discussion in above section, we know

$$\text{Range}(S) \supset \bigcup_{n=1}^{\infty} \text{Ker}(A_0^n).$$

Now, under our assumption $\text{rank}(S) < \infty$, and from our discussion in above section, we know when we restrict $A_0$ to range of $S$ is just a finite dimension operator with eigenvalue 0. With the help of Jordan canonical form, the equality in the relation 5.2 holds. Also from above discussion, when we restrict $A_0$ to $\text{Range}(S)^\perp$, 0 is not in the spectrum of it. So $(A_0 + S)$ has a bounded inverse. Hence, $A(z) + S$ also has a bounded inverse for $z$ small enough just like we discuss in section 3, as can be verified by Neuman series. So $B(z)$ is well-defined for $z$ small enough and bounded.
Moreover, if $A_0$ is self-adjoint, then from the results of above section, we have

$$S - S(A_0 + S)^{-1}S = 0.$$ 

Therefore, we get $B(z) = O(1)$ as $z \to 0$. Now suppose $B(z)$ is invertible on $S\mathcal{H}$. Let $T(z)$ be the right-hand side of 5.1. Then we just simply compute the following equalities,

$$T(z)A(z) = A(z)T(z) = I + \frac{1}{z}SB(z)^{-1}S(A(z) + S)^{-1} - S(A(z) + S)^{-1}^{-1}$$

$$- \frac{1}{z}S(A(z) + S)^{-1}SB(z)^{-1}S(A(z) + S)$$

$$= I + \frac{1}{z}SB(z)^{-1}S(A(z) + S)^{-1} - S(A(z) + S)^{-1}^{-1}$$

$$- \frac{1}{z}(S - zB(z))B(z)^{-1}S(A(z) + S)^{-1} = I.$$

For the other direction, suppose $A(z)$ is invertible. We define

$$D(z) = z(S + SA(z)^{-1}S = z(A(z) + S)[A(z)^{-1} - (A(z) + S)^{-1}](A(z) + S).$$

Then

$$B(z)D(z) = D(z)B(z)$$

$$= S + SA(z)^{-1}S - S(A(z) + S)^{-1}S - S(A(z) + S)^{-1}SA(z)^{-1}S$$

$$= S + S(A(z) + S)^{-1}SA(z)^{-1}S - S(A(z) + S)^{-1}SA(z)^{-1}S$$

$$= S.$$

In the context of Schrödinger operators, we consider $H_V = -\Delta + V$, where $V$ is a rapidly decaying potential. We set $H_0 = -\Delta$, then we can use the above lemma to study the expansion of the resolvent of $H_V$ from the expansion of the resolvent of $H_0$ in order to study the dispersive decay. One can find some details of this study in [ES]. In the investigation of [ES], the authors apply the above lemma twice to obtain an asymptotic expansion of $R_V$ near 0 with 0 is an eigenvalue or resonance.

6. Acknowledgment

I want to thank Prof. Wilhelm Schlag for suggesting the topics discussed in this note and some related reading materials. I also want to thank Jon Rubin for some interesting discussion. Finally, I would like to thank Ben Seeger for reading through the paper and helping me finalize it.

References


E-mail address: gc@math.uchicago.edu

Department of Mathematics, The University of Chicago, 5734 South University Avenue, Chicago, IL 60615, U.S.A