OVERCONVERGENT MODULAR FORMS AND THE FONTAINE-MAZUR CONJECTURE

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Abstract. Overconvergent modular forms are p-adic generalizations of classical modular forms. In [Ki03], Kisin proves that the p-adic Galois representation associated to an overconvergent modular form of finite slope has a crystalline period on which Frobenius acts as multiplication by the coefficient a_p. In this topic proposal, we give an exposition of Kisin's theorem and its application to the Fontaine-Mazur conjecture.

1. Introduction

1.1. **Representations of** $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$**.** Choose a prime number p. For objects appearing in arithmetic geometry one can often attach a continuous Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with coefficients in $\overline{\mathbb{Q}}_p$. For example, if E is an elliptic curve defined over $\mathbb{Q}$, then $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the Tate module $\varprojlim E[p^n](\overline{\mathbb{Q}})$, giving rise to a representation $\rho_E : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_p)$.

For each prime l, fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_l$; this induces an injection $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, whose image is the decomposition group D_l at l. The kernel of the action of D_l on the residue field $\overline{\mathbb{F}}_l$ is called the inertia group at l, which is denoted I_l. By virtue of the definition we have $D_l/I_l \cong \mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_l)$, which is naturally isomorphic to $\widehat{\mathbb{Z}}$. Any choice of a lift Frob_l of 1 to D_l is called an *arithmetic Frobenius at* l. A representation ρ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is said to be *unramified at* l if its restriction to D_l factors mod I_l. It is then determined by $\rho(\mathrm{Frob}_l)$.

As another example of a Galois representation, let $f = \sum_{n\geq 1} a_n q^n$ be a normalized eigenform of tame level[1] N, nebentypus ε and weight $k \geq 1$. Deligne [De71] and Deligne-Serre [DS74] attach to f a semisimple Galois representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$. It is characterised by the condition that for any prime $l \nmid pN$ it is unramified at l with the characteristic polynomial of $\rho_f(\mathrm{Frob}_l)$ being equal to $X^2 - a_l X + \varepsilon(l) l^{k-1}$.

1.2. **The Fontaine-Mazur conjecture.** In both of the examples presented so far, the attached representation is unramified at almost all primes, and its restriction to D_p is potentially semistable (a condition we shall define in section 2). By the work of many people, these properties are known to hold in general for representations attached to objects in arithmetic geometry which are defined over $\mathbb{Q}$. Fontaine and Mazur conjectured in [FM95] that the converse should also hold, namely, that these properties *characterise* representations coming from geometry. More precisely, the conjecture can be stated as follows.

Conjecture 1.1. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be an irreducible representation which is unramified outside finitely many primes, and whose restriction to D_p is potentially semistable. Then up to a twist, ρ is a subquotient of an étale cohomology group of an algebraic variety defined over $\mathbb{Q}$.*

For example, if $E = E_{/\mathbb{Q}}$ is an elliptic curve then ρ_E is the twist of $\mathrm{H}^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ by the cyclotomic character.

[1] By which we mean f is of level $\Gamma_1(Np^m)$ for some $m \geq 1$.

1.3. **Overconvergent modular forms.** Let $f = \sum_{n\geq 1} a_n q^n$ with be a normalised, cuspidal, overconvergent, p-adic modular eigenform of finite slope, which is of tame level N, nebentypus ε and weight κ (see section 3.3). The q-expansion of f is the p-adic limit of a sequence of q-expansions of classical eigenforms, and one can attach to f a Galois representation ρ_f of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by taking the limit of the Galois representations attached to forms in the sequence. The representation ρ_f is characterised by the condition that for any prime $l \nmid pN$ it is unramified at l with the characteristic polynomial of $\rho_f(\mathrm{Frob}_l)$ being equal to $X^2 - a_l X + \varepsilon(l)\kappa(l)l^{-1}$. It is unramified outside finitely many primes, but its restriction to D_p is usually not potentially semistable.

In [Ki03], Kisin proves most cases of the Fontaine-Mazur conjecture for the representations ρ_f (see section 4). This is a consequence of a representation-theoretic result he proves on $\rho_f|_{D_p}$. The statement of this result requires the language of p-adic Hodge theory, which we will recall with more details in section 2. Fontaine defines a ring $\mathrm{B}^+_{\mathrm{cris}}$ which is equipped with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) = D_p$ and a certain Frobenius operator φ. If V is a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $\overline{\mathbb{Q}}_p$, then $\mathrm{D}^+_{\mathrm{cris}}(V) = (\mathrm{B}^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is a vector space of finite dimension over $\mathbb{Q}_p$ equipped with an operator φ induced by the operator on $\mathrm{B}^+_{\mathrm{cris}}$. Kisin's theorem (Theorem 6.3 in [Ki03]) can be stated as follows.

Theorem 1.1. $\mathrm{D}^+_{\mathrm{cris}}\left(\rho_f|^{\vee}_{D_p}\right)^{\varphi=a_p} \neq 0.$

Remark 1.1. Let $E \subset \overline{\mathbb{Q}}_p$ be the field generated by the coefficients of f. Then Theorem 1.1 is equivalent to the statement that there exists a nonzero, E-linear, $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant map

$$\rho_f|_{D_p} \to \left(\mathrm{B}^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E\right)^{\varphi=a_p},$$

which is saying that the p-adic Galois representation $\rho_f|_{D_p}$ has a crystalline period on which Frobenius acts as multiplication by a_p.

Remark 1.2. Suppose f is a classical eigenform, but is of level $\Gamma_1(N)$, rather than being of tame level N. Then f has two corresponding oldforms in $\Gamma_1(Np)$, whose a_p-coefficients are the two roots of the Hecke polynomial $X^2 - a_p X + \varepsilon(p)p^{k-1}$. Thus if λ is a root of this polynomial, Theorem 1.1 implies that $\mathrm{D}^+_{\mathrm{cris}}\left(\rho_f|^{\vee}_{D_p}\right)^{\varphi=\lambda} \neq 0$.

The goal of this paper is to explain Theorem 1.1 and the ingredients of its proof, as well as its application to the Fontaine-Mazur conjecture.

1.4. **Notations and conventions.** The field K denotes a finite extension of $\mathbb{Q}_p$, whose absolute Galois group is G_K and whose maximal unramified subextension is K_0.

The cyclotomic character $\chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is the character given by the action of elements of $G_{\mathbb{Q}_p}$ on μ_{p^∞}, the roots of unity whose order is a power of p. It induces an isomorphism of $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ with $\mathbb{Z}_p^\times$. Its kernel is $H = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{p^\infty}))$, and $G_{\mathbb{Q}_p}/H = \Gamma$.

When given a modular form (p-adic or classical), we will sometimes assume it has coefficients in $\mathbb{Q}_p$ to simplify notations and arguments, although this is not necessary. Similarly, we will sometimes assume the residual representation has coefficients in $\mathbb{F}_p$.

All the characters and representations appearing in this paper are assumed to be continuous.

2. p-adic Galois Representations

2.1. **Sen's theory.** Let V be a finite dimensional representation of $G_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$. Sen's theory allows one to attach to V certain canonical invariants, called the *generalized Hodge-Tate weights* of V. He defines in [Se80] a certain linear operator Θ_V which acts on a finite dimensional $\mathbb{Q}_p$-vector space. Its characteristic polynomial $P_V(T) \in \mathbb{Q}_p[T]$ is called the *Sen polynomial* of V. We say V is *generalized Hodge-Tate* if Θ_V is semisimple. In that case, its eigenvalues $n_i \in \mathbb{C}_p$ (which are the roots of P_V) are called the *generalized Hodge-Tate weights* of V; this is equivalent to the existence of a $G_{\mathbb{Q}_p}$-equivariant isomorphism $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \oplus_i \mathbb{C}_p(n_i)$. If the n_i are all *integers,* we say that V is *Hodge-Tate* and the n_i are called the *Hodge-Tate weights* of V.

Example 2.1. 1. For any $n \in \mathbb{Z}$, the character χ^n is Hodge-Tate with Hodge-Tate weight n.

2. If $n \equiv 0 \mod p-1$, this can be interpolated to $\mathbb{Z}_p$ as follows. Let $\chi_2 = \chi^{p-1}$, which attains values in $1 + p\mathbb{Z}_p$. For any $s \in \mathbb{Z}_p$, the character χ_2^s is well defined and is generalized Hodge-Tate, with weight $(p-1)s$. It is Hodge-Tate if and only if $(p-1)s \in \mathbb{Z}$.

3. If f is a (classical) eigenform of weight $k \geq 1$ and whose coefficients lie in $\mathbb{Q}_p$, then the representation ρ_f is Hodge-Tate and its weights are 0 and $k-1$.

2.2. **Fontaine's rings of periods.** The goal of Fontaine's theory is to classify p-adic representations using objects of linear algebra. For this purpose he has constructed a number of *period rings*, all of which are $\mathbb{Q}_p$-topological algebras, endowed with a continuous action of $G_{\mathbb{Q}_p}$ and possibly some additional structure. The constructions of these rings are complicated, but for us it will be sufficient to recall some of their properties.

First, there is a ring $\mathrm{B}_{\mathrm{dR}}^+$, which is a discrete valuation ring with residue field $\mathbb{C}_p$ and whose maximal ideal is generated by a canonical element t. This element is a period for the cyclotomic character, meaning that $g(t) = \chi(g)t$ for $g \in G_{\mathbb{Q}_p}$. The ring $\mathrm{B}_{\mathrm{dR}}^+$ is then filtered by the ideals $\mathrm{Fil}^k \mathrm{B}_{\mathrm{dR}}^+ = t^k \mathrm{B}_{\mathrm{dR}}^+$ for $k \geq 0$, and its fraction field $\mathrm{B}_{\mathrm{dR}} = \mathrm{B}_{\mathrm{dR}}^+[1/t]$ has an induced filtration given by $\mathrm{Fil}^k \mathrm{B}_{\mathrm{dR}} = t^k \mathrm{B}_{\mathrm{dR}}^+$ for $k \in \mathbb{Z}$. Next, there is a ring B_{st} which has a Frobenius operator φ and a monodromy operator N, which satisfy $N\varphi = p\varphi N$, and which are compatible with the $G_{\mathbb{Q}_p}$-action. It contains the subring $\mathrm{B}_{\mathrm{cris}} = \mathrm{B}_{\mathrm{st}}^{N=0}$ and $\widehat{\mathbb{Q}_p^{\mathrm{un}}} \subset \mathrm{B}_{\mathrm{cris}}$. This gives rise to an injection $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p^{\mathrm{un}}} \mathrm{B}_{\mathrm{st}} \hookrightarrow \mathrm{B}_{\mathrm{dR}}$. The ring $\mathrm{B}_{\mathrm{cris}}^+$ is defined as $\mathrm{B}_{\mathrm{cris}} \cap \mathrm{B}_{\mathrm{dR}}^+$. The filtration of B_{dR} induces filtrations on $\mathrm{B}_{\mathrm{st}}, \mathrm{B}_{\mathrm{cris}}$ and $\mathrm{B}_{\mathrm{cris}}^+$.

Remark 2.1. A technical issue we should point out is that the inclusion $\mathrm{B}_{\mathrm{cris}}^+ \subset \mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}}$ is *not* an equality. However, due to a theorem of Fontaine, it is true that $\left(\mathrm{B}_{\mathrm{cris}}^+\right)^{\varphi=1} = \left(\mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}}\right)^{\varphi=1} = \mathbb{Q}_p$. More generally, if M is a finite-dimensional $\mathbb{Q}_p$-vector space endowed with a Frobenius φ, one has $\left(\mathrm{B}_{\mathrm{cris}}^+ \otimes_{\mathbb{Q}_p} M\right)^{\varphi=1} = \left(\mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} M\right)^{\varphi=1}$ (see Lemma 3.2 in [Ki03]).

2.3. **p-adic Hodge theory.** If V is a p-adic Galois representation of $G_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ and $* \in \{\mathrm{dR}, \mathrm{st}, \mathrm{cris}\}$, let $\mathrm{D}_*(V) = (\mathrm{B}_* \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$ (resp. $\mathrm{D}_*^+(V) = (\mathrm{B}_*^+ \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$). Then $\mathrm{D}_*(V)$ and $\mathrm{D}_*^+(V)$ are vector spaces over $\mathbb{Q}_p$ of dimension $\leq \dim_{\mathbb{Q}_p}(V)$. We say V is *de Rham* (resp. *semistable, crystalline*) if $\dim_{\mathbb{Q}_p} \mathrm{D}_*(V) = \dim_{\mathbb{Q}_p}(V)$ for $* = \mathrm{dR}$ (resp. st,cris). There are implications

$$\text{crystalline} \Longrightarrow \text{semistable} \Longrightarrow \text{de Rham} \Longrightarrow \text{Hodge-Tate}.$$

The vector spaces $\mathrm{D}_*(V)$ inherit additional structure from that of B_*. Namely, the space $\mathrm{D}_{\mathrm{dR}}(V)$ is a filtered vector space, the space $\mathrm{D}_{\mathrm{st}}(V) \subset \mathrm{D}_{\mathrm{dR}}(V)$ is a filtered (φ, N)-module, and the space $D_{\mathrm{cris}}(V) = \mathrm{D}_{\mathrm{st}}(V)^{N=0}$ is a filtered φ-module.

To a filtered (φ, N)-module D one can associate two polygons. The *Hodge polygon* $P_H(D)$, whose slopes have lengths according to the jumps in the filtration; and the *Newton polygon* $P_N(D)$, whose slopes match the slopes of φ. We say D is *admissible* if the endpoints of $P_H(D)$ and $P_N(D)$ are the same and if $P_H(D')$ lies below $P_N(D')$ for every subobject D' of D. One can show that $\mathrm{D}_{\mathrm{st}}(V)$ and $\mathrm{D}_{\mathrm{cris}}(V)$ are admissible.

The functors $\mathrm{D}_*(V)$ and $\mathrm{D}_*^+(V)$ can be generalized to the case where $K/\mathbb{Q}_p$ is a finite extension and V is a representation of G_K over $\mathbb{Q}_p$, and similar properties hold. However, while $\mathrm{D}_{\mathrm{dR}}(V)$ is a

K-vector space induced with a filtration, $\mathrm{D}_{\mathrm{cris}}(V)$ and $\mathrm{D}_{\mathrm{st}}(V)$ are vector spaces over K_0, so their filtration is only defined after extending scalars to K. A representation V of $G_{\mathbb{Q}_p}$ is then called *potentially semistable* if for some finite extension K of $\mathbb{Q}_p$, the representation $V|_{G_K}$ is semistable for G_K.

2.4. **Kisin's theorem for classical modular forms.** Let V be a potentially semistable p-adic Galois representation of $G_{\mathbb{Q}_p}$. We set

$$\mathrm{D}_{\mathrm{pst}}(V) = \operatorname*{colim}_{[K:\mathbb{Q}_p]<\infty} (\mathrm{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

The $\mathbb{Q}_p^{\mathrm{un}}$-vector space $\mathrm{D}_{\mathrm{pst}}(V)$ is endowed with an action of a semilinear Frobenius φ, a monodromy operator N and a $G_{\mathbb{Q}_p}$-action. The *local L-factor* of the representation V is the rational function

$$L(V,T) = \det\left(1 - \left(\mathrm{Frob}_p^{-1}\varphi\right) T|\mathrm{D}_{\mathrm{pst}}(V)^{I_p,N=0}\right)^{-1}.$$

Lemma 2.1. *If $\alpha \in \mathbb{Q}_p$ is an eigenvalue of $\mathrm{Frob}_p^{-1}\varphi$ acting on $\mathrm{D}_{\mathrm{pst}}(V^\vee)^{I_p,N=0}$, then $\mathrm{D}_{\mathrm{cris}}(V^\vee)^{\varphi=\alpha} \neq 0$.*

Proof. By the theory of Galois descent, we have

$$\mathrm{D}_{\mathrm{cris}}(V^\vee) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{un}} = \mathrm{D}_{\mathrm{st}}(V^\vee)^{N=0} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{un}} \xrightarrow{\sim} \mathrm{D}_{\mathrm{pst}}(V^\vee)^{I_p,N=0},$$

so $\mathrm{D}_{\mathrm{cris}}(V^\vee)$ is a $\mathbb{Q}_p$-form of $\mathrm{D}_{\mathrm{pst}}(V^\vee)^{I_p,N=0}$. The operator $\mathrm{Frob}_p^{-1}\varphi$ acts linearly on $\mathrm{D}_{\mathrm{pst}}(V^\vee)^{I_p,N=0}$ with eigenvalue $\alpha \in \mathbb{Q}_p$, so it must have an eigenvector in $\mathrm{D}_{\mathrm{cris}}(V^\vee)$. On the other hand, $\mathrm{D}_{\mathrm{cris}}(V^\vee)$ is fixed by the $G_{\mathbb{Q}_p}$ action, so $\mathrm{Frob}_p^{-1}\varphi$ acts on $\mathrm{D}_{\mathrm{cris}}(V^\vee)$ by φ. □

We are now ready to prove that Kisin's theorem holds for classical modular forms. To fix notation, let f be a normalized eigenform of tame level N and let $V = V_f$ be the attached Galois representation. To simplify the notations and the arguments, we will assume V_f has coefficients in $\mathbb{Q}_p$, but it should be noted this assumption is not essential. The key input is a local-global compatibility result of Saito.

Proposition 2.1. *Theorem 1.1 holds if f is classical.*

Proof. By the main result of [Sa97], the automorphic and Galois L-factors of f at p coincide.[2] Namely,

$$\det\left(1 - \left(\mathrm{Frob}_p^{-1}\varphi\right) T|\mathrm{D}_{\mathrm{pst}}(V)^{I_p,N=0}\right)^{-1} = (1 - a_p T)^{-1}.$$

By the assumption $a_p \neq 0$, the equation implies that a_p is an eigenvalue for the action of $\mathrm{Frob}_p^{-1}\varphi$ on $\mathrm{D}_{\mathrm{pst}}(V^\vee)^{I_p,N=0}$. Therefore, $\mathrm{D}_{\mathrm{cris}}(V^\vee)^{\varphi=a_p} \neq 0$ by Lemma 2.1.
On the other hand, the Hodge-Tate weights of $V^\vee$ are 0 and $1-k$ by Example 2.1.3, which are nonpositive. This means that the de Rham filtration of $\mathrm{D}_{\mathrm{cris}}(V^\vee)$ is concentrated at nonpositive degrees, so

$$\mathrm{D}_{\mathrm{cris}}(V^\vee) = \left(\mathrm{Fil}^0\mathrm{B}_{\mathrm{cris}} \otimes V^\vee\right)^{G_{\mathbb{Q}_p}}.$$

Finally, the φ-eigenvalues of $\left(\mathrm{Fil}^0\mathrm{B}_{\mathrm{cris}} \otimes V\right)^{G_{\mathbb{Q}_p}}$ and $\left(\mathrm{B}_{\mathrm{cris}}^+ \otimes V\right)^{G_{\mathbb{Q}_p}}$ coincide by remark 2.1. □

[2] In fact, Saito proves a stronger result. Namely, if $\pi_p(f)$ is the smooth admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation associated to f, then the Weil-Deligne representation attached to it by the local Langlands correspondence coincides with the Weil-Delgine representation attached to V_f. In particular, their L-factors are the same.

2.5. **Families of Galois representations.** Recall that for a p-adic field E, the n-variable *Tate algebra* over E is

$$E\langle X_1, ..., X_n\rangle = \left\{\sum a_J X^J : |a_J| \to 0 \text{ as } \|J\| \to \infty\right\}.$$

An *affinoid algebra* over E is then any quotient of some $E\langle X_1, ..., X_n\rangle$. To an affinoid algebra R one can associate an *affinoid* $\mathrm{Sp}(R)$, a p-adic geometric object, in much the same way that to a ring R one can associate the affine scheme $\mathrm{Spec}(R)$. For example, the algebra $E\langle X_1, ..., X_n\rangle$ is thought of as the ring of functions on $\mathrm{Sp}(E\langle X_1, ..., X_n\rangle)$, the closed n-dimensional polydisc over E. Affinoids can be glued to form the more general *rigid analytic spaces*, in the same way affine schemes can be glued to form schemes.

By a *family of Galois representations* over an analytic space X we shall mean a free $\mathcal{O}_X$-module V, equipped with a continuous $G_{\mathbb{Q}_p}$-action. For every point $x \in X$, we obtain a p-adic Galois representation V_x over the residue field $k(x)$; thus the points of X parametrize a family of p-adic Galois representations. We shall see examples of such families in the next section.

The constructions of 2.1 and 2.3 can be interpolated to families (for instance, see [BC08]). For each point $x \in X$, there is a Sen operator $\Theta_{V_x} \in \mathrm{End}_{k(x)}(V_x)$, and this family interpolates to give an operator $\Theta_V \in \mathrm{End}_{\mathcal{O}_X}(V)$. Consequently, the various Sen polynomials interpolate to a polynomial with coefficients which are rigid analytic functions on X. In a similar way, the functors of p-adic Hodge theory can be interpolated to families. For example, if $X = \mathrm{Sp}(R)$ is an affinoid, one can consider

$$\mathrm{D}^+_{\mathrm{cris}}(V) := \left(\left(\mathrm{B}^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p} R\right) \otimes_R V\right)^{G_{\mathbb{Q}_p}},$$

which is a finite filtered φ-module over R. Specializing to a point x then gives the usual $\mathrm{D}^+_{\mathrm{cris}}(V_x)$ for the p-adic Galois representation V_x.

3. Finite slope subspaces

3.1. **The X_{fs} construction.** Fix a separated rigid analytic space X and a family of Galois representations V over X. At the heart of the interpolation argument which allows Kisin to extend the results on classical modular forms to overconvergent modular forms is the construction of the finite slope subspace X_{fs}.

The statement (Proposition 5.4 in [Ki03]) requires the introduction of two technical notions; for the precise definitions we refer the reader to section 5 of [Ki03]. An open set[3] $U \subset X$ is *scheme-theoretically dense* in X if it is locally the analytification of a dense, open Zariski set. Given an invertible analytic function Y on X, a map $X' \to X$ of analytic spaces is called *Y-small* if, roughly, after pulling it back to X' it is constant up to adding a nilpotent function. Suffices to say that in all of our subsequent applications all our maps will be Y-small, and that in any case this assumption has been removed by the work of Liu in [Li15] and Kedlaya, Pottharst and Xiao in [KPX14]. We denote by $V^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(V, \mathcal{O}_X)$ the dual $\mathcal{O}_X$-module with the induced $G_{\mathbb{Q}_p}$-action.

Theorem 3.1. *Let Y be an invertible analytic function on X. Suppose that the Sen polynomial $P_V(T) = TQ(T)$ for some $Q \in \mathcal{O}_X(X)[T]$. Then there exists a unique Zariski closed subspace $X_{\mathrm{fs}} = X_{\mathrm{fs}}(Y) \subset X$ such that*

(1) For every integer $j \geq 0$ the subspace $X_{\mathrm{fs},Q(j)}$ is scheme-theoretically dense in X_{fs}.[4]

[3]More precisely, an admissible open set

[4]Note that our normalization for this section is the negative of the one used in [Ki03].

(2) For any affinoid algebra and any Y-small map $f : \mathrm{Sp}(R) \to X$ which factors through $X_{Q(j)}$ for each integer $j \geq 0$, the map f factors through X_{fs} if and only if

$$\mathrm{D}^+_{\mathrm{cris}}(f^*V^\vee(\mathrm{Sp}(R)))^{\varphi=Y} \xrightarrow{\sim} \mathrm{D}^+_{\mathrm{dR}}(f^*V^\vee(\mathrm{Sp}(R))).$$

Note that if $X = \mathrm{Sp}(R)$ is an affinoid (so that V can be thought of as a module over R), and $f : \mathrm{Sp}(R') \to X$ is a Y-small map, the condition above simplifies to

$$\mathrm{D}^+_{\mathrm{cris}}(V^\vee \otimes_R R')^{\varphi=Y} \xrightarrow{\sim} \mathrm{D}^+_{\mathrm{dR}}(V^\vee \otimes_R R').$$

Remarks on the proof. The proof consists of several steps. First one proves that X_{fs} is unique, an argument which is geometric in nature. The second step is to reduce to the case where X is an affinoid, by showing the characterisation of X_{fs} is compatible with coverings. This step is not formal, and some of the arguments require the use of Sen's theory in families. The third step, which is perhaps the most involved one, is to construct X_{fs} in the case that $X = \mathrm{Sp}(R)$ is an affinoid. One particular complication is that $\mathrm{B}^+_{\mathrm{dR}}$ is not a Banach algebra, and the condition specifing the map from $\mathrm{D}^+_{\mathrm{cris}}(V^\vee)^{\varphi=Y}$ to $\mathrm{D}^+_{\mathrm{dR}}(V^\vee)$ is an isomorphism cannot a-priori be defined by requiring analytic functions to vanish. This situation is remedied by showing that any such map has to factor through

$$\mathrm{D}^{+,k}_{\mathrm{dR}}(V^\vee) := \left(\left(\left(\mathrm{B}^+_{\mathrm{dR}}/t^k\mathrm{B}^+_{\mathrm{dR}}\right)\widehat{\otimes}_{\mathbb{Q}_p}R\right)\otimes_R V^\vee\right)^{G_{\mathbb{Q}_p}}$$

for k large enough, and noting that $\mathrm{B}^+_{\mathrm{dR}}/t^k\mathrm{B}^+_{\mathrm{dR}}$ is a Banach algebra. Most of the work done in sections 2,3 and 4 of [Ki03] is required for this argument.

3.2. **Example: deformations in dimension 1.** One way to construct families of Galois representations is by the use of universal deformation rings. Recall that if $\overline{\rho} : G \to \mathrm{GL}_n(\mathbb{F}_p)$ is a representation of a group G, a *deformation* of $\overline{\rho}$ to a complete Noetherian local $\mathbb{Z}_p$-algebra A with residue field $\mathbb{F}_p$ is a lifting[5] $\rho : G \to \mathrm{GL}_n(A)$. If G satisfies a certain finiteness condition and if $\overline{\rho}$ is absolutely irreducible, both of which will hold in all of our future discussions, there exists a *universal deformation ring* $R_{\overline{\rho}}$ with a deformation $\rho_{\mathrm{univ}} : G \to \mathrm{GL}_n(R_{\overline{\rho}})$ whose A-valued points (for A as above) parametrize all deformations of $\overline{\rho}$ to A. The rigid analytic space[6] $X_{\overline{\rho}} = R_{\overline{\rho}}[1/p]^{\mathrm{rig}}$ then parametrizes deformations[7] of $\overline{\rho}$ to $\overline{\mathbb{Z}}_p$ and so gives a family of p-adic Galois representations for $G = G_{\mathbb{Q}_p}$.

As the simplest possible example, consider the trivial representation $\overline{\rho} = \overline{\rho}_{\mathrm{triv}} : G_{\mathbb{Q}_p} \to \mathbb{F}_p^\times$. In this case the universal deformation ring is $R_{\overline{\rho}} = \mathbb{Z}_p[[G^{\mathrm{ab},p}_{\mathbb{Q}_p}]]$, where the group $G^{\mathrm{ab},p}_{\mathbb{Q}_p}$ is the maximal abelian and pro-p quotient of $G_{\mathbb{Q}_p}$. By class field theory $G^{\mathrm{ab},p}_{\mathbb{Q}_p} \cong (1+p\mathbb{Z}_p) \times \mathbb{Z}_p$, the $(1+p\mathbb{Z}_p)$ corresponding to the image of Γ under χ and the $\mathbb{Z}_p$ being generated by the image of a Frobenius element σ. Thus

$$X_{\overline{\rho}} = \mathbb{Z}_p[[G^{\mathrm{ab},p}_{\mathbb{Q}_p}]][1/p]^{\mathrm{rig}} \cong \mathbb{Z}_p[[(1+p\mathbb{Z}_p)\times\mathbb{Z}_p]][1/p]^{\mathrm{rig}} \cong \mathbb{Z}_p[[X,Y]][1/p]^{\mathrm{rig}}$$

is a 2-dimensional open polydisc, with coordinate functions given by a generator $\gamma \in \Gamma$ and by σ.

It is not too hard to check that for this family of p-adic Galois representations, the Sen polynomial of $X_{\overline{\rho}}$ is given by $T - \frac{\log(\gamma)}{\log(\chi(\gamma))}$. We cannot apply the X_{fs} construction to $X_{\overline{\rho}}$, as the condition that this polynomial is divisible by T is not satisfied. However if we let $X = \{\log\gamma = 0\} \subset X_{\overline{\rho}}$ be the vanishing locus of $\log(\gamma)$, the Sen polynomial of X is given by $P(T) = T$ and the space X_{fs} is

[5] More precisely, it is an equivalence class of such liftings up to a conjugation.

[6] If $R_{\overline{\rho}}[1/p] = \mathbb{Z}_p[[X_1,...,X_n]]/(f_1,...,f_m)[1/p]$, this can be thought of as the vanishing locus of $f_1,...,f_m$ in the open unit n-polydisc.

[7] More precisely, if x is a point of $X_{\overline{\rho}}$ whose residue field is E, then x gives rise to a representation $\rho_x : G \to \mathrm{GL}_n(\mathcal{O}_E)$ over the ring of integers $\mathcal{O}_E$ given by specializing ρ_{univ} and whose reduction gives the base change of $\overline{\rho}$ to the residue field of E.

defined. A point $x \in X$ is given by the data of a character $\eta_x : G_{\mathbb{Q}_p} \to E^\times$ for a finite extension $E/\mathbb{Q}_p$. Then condition 2 is saying that $x \in X_{\mathrm{fs}}$ if and only if

$$\mathrm{D}_{\mathrm{cris}}^{+}(E(\eta_x^{-1}))^{\varphi=Y} \xrightarrow{\sim} \mathrm{D}_{\mathrm{dR}}^{+}(E(\eta_x^{-1})).$$

One then computes that $\mathrm{D}_{\mathrm{dR}}^{+}(E(\eta_x^{-1}))$ is always a 1-dimensional E-vector space. On the other hand, $\mathrm{D}_{\mathrm{cris}}^{+}(E(\eta_x^{-1}))$ is nonzero if and only if η_x is unramified, and in that case φ acts on it by $\eta_x(\sigma)$. Thus

$$X_{\mathrm{fs}} = \{\gamma = 1\} \cap \{Y = \sigma\} \subset \{\log \gamma = 0\} = X.$$

3.3. **The eigencurve.** Fix an integer $N \geq 1$ prime to p and let S be a finite set of primes containing the primes dividing pN and ∞. Let $\mathbb{Q}_S$ be the maximal extension of $\mathbb{Q}$ which is unramified outside S and let $\overline{\rho} : \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_p)$ be an odd, absolutely irreducible representation. Let $X_{\overline{\rho}} = R_{\overline{\rho}}[1/p]^{\mathrm{rig}}$ be the rigid analytic space associated to $\overline{\rho}$, so that each point $x \in X_{\overline{\rho}}$ corresponds to a representation $\rho : \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}) \to \mathrm{GL}_2\left(\overline{\mathbb{Z}}_p\right)$.

By the work of Coleman and Mazur in [CM98], the points $(x, \lambda) \in X_{\overline{\rho}} \times \mathbb{G}_m(\overline{\mathbb{Q}}_p)$ such that V_x is attached to a cuspidal eigenform of level $\Gamma_1(Np^m)$ with $m \geq 1$ and $a_p(f) = \lambda$ interpolate to a rigid analytic curve $\mathcal{C}$, called *the eigencurve*. The $\mathbb{C}_p$-valued points of $\mathcal{C}$ correspond to normalized finite slope p-adic overconvergent cuspidal eigenforms of tame level N and whose attached residual representation $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}) \to \mathrm{GL}_2\left(\overline{\mathbb{F}}_p\right)$ is $\overline{\rho}$. One then obtains a free $\mathcal{O}_{\mathcal{C}}$-module V of rank 2 equipped with a $G_{\mathbb{Q}_p}$ action, such that for any point $x \in \mathcal{C}(\mathbb{C}_p)$ corresponding to an overconvergent eigenform f, there is a natural isomorphism between V_f and V_x. In other words, V is a family of p-adic Galois representations over $\mathcal{C}$ which parametrizes representations attached to overconvergent modular forms.

As the classical eigenforms of tame level N vary, their weights vary in $\mathbb{Z}$. Accordingly as these modular forms interpolate to $\mathcal{C}$ their weights interpolate (in a suitable sense) to a rigid analytic curve $\mathcal{W}$, called the *weight space*. Its $\mathbb{C}_p$-valued points are given by

$$\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}((\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times, \mathbb{C}_p^\times),$$

consisting of the *weight-character* of the overconvergent forms. For example, suppose $N = 1$ and f is a classical modular form of level $\Gamma_1(p^m)$ and weight k, with character $\varepsilon : (\mathbb{Z}/p^m\mathbb{Z})^\times \to \mathbb{C}_p^\times$. Then its weight-character is $w = \lambda_k \varepsilon : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, where λ_k is the character $z \mapsto z^k$ and ε is thought of as a character of $\mathbb{Z}_p^\times$ by composing with reduction mod p^m. The association which sends an overconvergent modular form to its weight-character then gives a projection $w : \mathcal{C} \to \mathcal{W}$. The *weight* κ of an overconvergent form is then the unique $\kappa \in \mathbb{C}_p$ such that $\log_p(w(f)(1, z)) = \kappa \log_p(z)$ for any $z \in \mathbb{Z}_p^\times$, extending the notion of a classical weight.

Remark 3.1. By Theorem C of [CM98], this map is locally in the domain finite flat. In geometric terms, this means that $\mathcal{C} \to \mathcal{W}$ is locally a ramified covering of curves. This allows us to use p-adic approximation arguments, in the sense that if f is an overconvergent modular form which is unramified over the weight space and if κ' is a weight which is (p-adically) sufficiently close to $\kappa(f)$, then there exists an overconvergent modular form f' close to f with $\kappa(f') = \kappa'$. Indeed, a ramified covering admits a local section at unramified points.

3.4. **Kisin's theorem.** We are now ready to sketch the proof of Theorem 1.1. The idea is to deduce the statement for general overconvergent forms from the classical case using a p-adic interpolation argument, but the arguments are subtle. The proof can roughly be divided into 7 steps.

1. By Proposition 2.1 $\mathrm{D}_{\mathrm{cris}}^{+}\left(\rho_f|_{D_p}^{\vee}\right)^{\varphi=a_p} \neq 0$ if f is classical.

2. One verifies the property for overconvergent f with weight in $\mathbb{Z}_p$. Since $\mathbb{Z}$ is p-adically dense in $\mathbb{Z}_p$, Remark 3.1 implies that f can be approximated by forms with weight in $\mathbb{Z}$. In fact, any such form can be approximated by *classical* forms. This follows, for instance, from Coleman's theorem which we discuss in section 4.2. The property $\mathrm{D}^+_{\mathrm{cris}}\left(\rho_f|^\vee_{D_p}\right)^{\varphi=a_p} \neq 0$ follows by showing it is sufficiently "continuous"; for more details on this step, see Corollary 5.15 of [Ki03].

3. We now consider the family of Galois representations V over $\mathcal{C}$. The classical modular forms are Zariski dense in $\mathcal{C}$ and their associated representations have Hodge-Tate weights 0 and $k-1$, according to Example 2.1.3. Hence, the Sen polynomial of V is $P(T) = T(T-(\kappa-1))$, which is divisible by T. So Theorem 3.1 applies and the space $\mathcal{C}_{\mathrm{fs}}$ is defined.

4. With the notations of Theorem 3.1 we have $Q(T) = T-(\kappa-1)$. Let f be an overconvergent form with weight in $\mathbb{Z}_p - \mathbb{Z}$ whose corresponding point is $x : \mathrm{Sp}(E) \to \mathcal{C}$. Then $Q(j) \neq 0$ for each $j \geq 0$, so x factors through $\mathcal{C}_{Q(j)}$. By condition 2 of Theorem 3.1 and step 2, we see that $x \in \mathcal{C}_{\mathrm{fs}}$. Thus $\mathcal{C}_{\mathrm{fs}}$ contains all the overconvergent forms whose weight lies in $\mathbb{Z}_p - \mathbb{Z}$.

5. As $\mathcal{C}$ is the Zariski closure of the classical modular locus, any Zariski open set of $\mathcal{C}$ contains a classical point unramified over weight space. But such a point has weight in $\mathbb{Z}$, and the subspace $\mathcal{C}_{\mathrm{fs}}$ contains the points whose weight lies in $\mathbb{Z}_p - \mathbb{Z}$, so it intersects any Zariski open set, by Remark 3.1. This means that $\mathcal{C}_{\mathrm{fs}}$ is Zariski dense in $\mathcal{C}$.

6. By construction, the subspace $\mathcal{C}_{\mathrm{fs}} \subset \mathcal{C}$ is *Zariski* closed. By step 5, there must be an equality $\mathcal{C}_{\mathrm{fs}} = \mathcal{C}$.

7. One can independently show that for $f \in \mathcal{C}_{\mathrm{fs}}(\overline{\mathbb{Q}}_p)$ we have $\mathrm{D}^+_{\mathrm{cris}}\left(\rho_f|^\vee_{D_p}\right)^{\varphi=a_p} \neq 0$ (see Corollary 5.16 of [Ki03]). As $\mathcal{C}_{\mathrm{fs}} = \mathcal{C}$, we are done. □

4. Application to the Fontaine-Mazur conjecture

4.1. **The Fontaine-Mazur conjecture for odd, 2-dimensional representations.** Recall from section 1.2 that the Fontaine-Mazur conjecture predicts which p-adic representations V of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ should arise from a global geometric object. It says that such representations should be unramified except for finitely many primes and should be potentially semistable when restricted to the decomposition group D_p. If in addition $\dim(V) = 2$ and V is odd, then Fontaine and Mazur formulated the following more precise conjecture (conjecture 3c of [FM95]).

Conjecture 4.1. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be an odd, irreducible representation which is unramified outside finitely many primes and whose restriction to D_p is potentially semistable with distinct Hodge-Tate weights. Then V is the twist of a Galois representation associated to a modular form of weight $k \geq 2$.*

For example, if $E = E_{/\mathbb{Q}}$ is an elliptic curve then its Tate module ρ_E satisfies the above assumptions. Indeed, according to the modularity theorem, there exists a modular form f of weight 2 with $\rho_f = \rho_E$ and the conjecture is true in this case.

An application of Kisin's theorem answers the Fontaine-Mazur conjecture in the affirmative for representations coming from overconvergent forms, if a certain case is excluded. More precisely, an overconvergent form f is called *exceptional* if

(1) The representation $\rho_f|_{D_p}$ is the direct sum of two $\overline{\mathbb{Z}}_p$-valued characters, and
(2) The two characters are congruent modulo the maximal ideal $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$.

We then have the following (Theorem 6.6 (3) of [Ki03]).

Theorem 4.1. *Let f be a finite slope overconvergent eigenform such that ρ_f is potentially semistable when restricted to D_p and has distinct Hodge-Tate weights.*[8] *Assume that f is not exceptional. Then conjecture 4.1 holds for V_f.*

4.2. **Coleman's criterion.** To prove theorem 4.1, one needs to relate the potential semistablity of $\rho_f|_{D_p}$ to classical modular forms. Kisin does this by showing that if f has weight $k \geq 2$ then it is actually classical unless $\rho_f|_{D_p}$ splits. This is achieved by the use of Coleman's classicality criterion for overconvergent modular forms in [Col97].

To state the result, we need to introduce the Θ operator. It acts on modular forms by the formula

$$\Theta\left(\sum_{n\geq 0} a_n q^n\right) = \sum_{n\geq 0} n a_n q^n,$$

mapping forms of weight k to forms of weight $k+2$. If $k \geq 1$, its iteration Θ^{k-1} maps overconvergent eigenforms of weight $2-k$ to overconvergent eigenforms of weight k.

Coleman's theorem can then be stated as follows.

Theorem 4.2. *Let $f = \sum_{n\geq 1} a_n q^n$ be an overconvergent eigenform of weight $k \geq 2$. Suppose that either*

1. $v_p(a_p) < k-1$, or

2. $v_p(a_p) = k-1$ and f is not in the image of Θ^{k-1}.

Then f is classical.

Remark 4.1. The effect of Θ^{k-1} on Galois representations is given by the twist

$$\rho_{\Theta^{k-1}f} \cong \chi^{k-1} \otimes \rho_f.$$

Indeed, if l is a prime not dividing pN, then we have

$$\mathrm{Tr}(\mathrm{Frob}_l|\rho_{\Theta^{k-1}f}) = l^{k-1}a_l = \chi^{k-1}(l)a_l = \mathrm{Tr}\left(\mathrm{Frob}_l|\chi^{k-1}\otimes\rho_f\right),$$

and the claimed isomorphism follows from the Chebotarev density theorem.

In particular, Conjecture 4.1 holds for ρ_f if and only if it holds for $\rho_{\Theta^{k-1}f}$.

4.3. **The Fontaine-Mazur conjecture for overconvergent modular forms.** Using Kisin's theorem and Theorem 4.2, we are in a position to prove Theorem 4.1. We shall need some notation. Let f be a finite slope overconvergent eigenform such that $\rho_f|_{G_{\mathbb{Q}_p}}$ is potentially semistable and has distinct Hodge-Tate weights. Further, let K be a finite Galois extension such that $\rho_f|_{G_K}$ is semistable. For simplicity, we shall assume the image of ρ_f is contained in $\mathrm{GL}_2(\mathbb{Q}_p)$ (this does not make much of a difference in terms of the arguments, but simplifies the notations considerably). Recall that our aim is to show that if f is not exceptional, then ρ_f is the twist of a Galois representation attached to a classical modular form of weight $k \geq 2$.

Proof of theorem 4.1. As $\rho_f|_{D_p}$ is potentially semistable, it is also Hodge-Tate. As we have seen in 3.4, these weights are 0 and $k-1$, so k must be an integer. Replacing f by $\Theta^{1-k}f$ if neccessary, Remark 4.1 allows us to reduce to the case $k \geq 2$.

As $\rho_f|_{G_K}$ is semistable, the associated (φ, N)-module $D = \mathrm{D}_{\mathrm{st}}\left(\rho_f|_{G_K}^{\vee}\right)$ has $\dim_{K_0} D = \dim_{\mathbb{Q}_p}\rho_f = 2$. The Hodge polygon $P_H(D)$ has 1 segment of slope 0 and 1 segment of slope $k-1$, because it is determined by the filtration, which in turn is determined by the Hodge-Tate weights. On the other hand, by Kisin's theorem the slope $v_p(a_p)$ appears in the Newton polygon $P_N(D)$. But D is

[8]By Example 2.1.3, this the same as requiring that the weight of f is different from 1.

admissible, so $P_N(D)$ lies above $P_H(D)$ and has the same endpoints. Whether $v_p(a_p)$ appears with multiplicity 1 or with multiplicity 2, it is always implied that $0 \leq v_p(a_p) \leq k-1$.

If $v_p(a_p) < k-1$ or $v_p(a_p) = k-1$ and f is not in the image of Θ^{k-1}, we are done by Coleman's theorem.

Otherwise, $f = \Theta^{k-1}g$ for some overconvergent form f. Twisting by χ^{k-1} and applying Kisin's theorem, we find that $(t^{k-1}\mathrm{B}^+_{\mathrm{cris}} \otimes \rho_f^\vee)^{G_K, \varphi=a_p} \neq 0$ (this is stronger than what Kisin's theorem would give when applied directly to ρ_f). As the Hodge-Tate weights of $\rho_f^\vee$ are 0 and $1-k$, one deduces the existence of an extension and of 1-dimensional $G_{\mathbb{Q}_p}$-representations over $\mathbb{Q}_p$,

$$0 \to \rho_1 \to \rho_f \to \rho_2 \to 0,$$

where ρ_1 has Hodge-Tate weight 0 and ρ_2 has Hodge-Tate weight $k-1$. However, it is well known that an extension with increasing Hodge-Tate weights can be de Rham only if it is split (see for instance the argument of Example 6.3.5 in [BC]). Such split representations are known to come from a modular form if $\overline{\rho}_1 \neq \overline{\rho}_2$ by the work of Skinner-Wiles in [SW99, SW01]. The condition $\overline{\rho}_1 \neq \overline{\rho}_2$ holds because f is not exceptional. □

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