THE p-ADIC SIMPSON CORRESPONDENCE FOR RIGID ANALYTIC VARIETIES

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Abstract. This is a very brief summary of some main parts of Yupeng Wang's paper on the p-adic Simpson correspondence for rigid analytic varieties, together with a few examples.

1. Introduction and notation

The purpose of this note is to summarize the paper "A p-adic Simpson correspondence for rigid analytic varieties" by Yupeng Wang. Throughout, we shall have the following **global** setting. We have a formally smooth formal scheme $\mathfrak{X}$ over $\mathrm{Spf}\mathcal{O}_C$ (here $C = \mathbb{C}_p$), which is locally of topologically finite type. We shall assume it is liftable, which means it has a formally smooth lifting to $\mathrm{Spf}\mathrm{A}_2$, where $\mathrm{A}_2 := \mathrm{A}_{\mathrm{inf}}(\mathcal{O}_C)/\xi^2$. We set X for the generic fiber of $\mathfrak{X}$ over $\mathrm{Spa}\mathbb{Q}_p$, which is a rigid analytic variety.

When specializing to a **local** setting, we shall always assume $\mathfrak{X} = \mathrm{Spf}R^+$ and admits toric coordinates, i.e. admits a formally etale morphism $\mathfrak{X} \hookrightarrow \mathrm{Spf}\mathcal{O}_C\left\langle T_1^{\pm 1}, ..., T_d^{\pm 1}\right\rangle$. The formal smoothness assumption implies that any global $\mathfrak{X}$ is locally in the etale topology in this local form. In this local situation, we set $R_\infty^+ = R^+ \otimes_{\mathcal{O}_C\left\langle T_1^{\pm 1},...,T_d^{\pm 1}\right\rangle} \mathcal{O}_C\left\langle T_1^{\pm(1/p^\infty)}, ..., T_d^{\pm(1/p^\infty)}\right\rangle$, which is affinoid perfectoid. Write $\widehat{R}_\infty = \widehat{R}_\infty^+[1/p]$; in this setting we also set $X_\infty = \mathrm{Spa}\left(\widehat{R}_\infty, \widehat{R}_\infty^+\right)$, which is a perfectoid space. It is a Galois covering of X with Galois group Γ_∞; it is an open subgroup of $\mathbb{Z}_p^d$.

Recall that we have the pro-etale site X_{proet}. It is defined in Scholze's p-adic Hodge theory paper, and its open subsets are roughly of the form $V \to U \to X$, where U is an etale morphism and $V \to U$ is an inverse limit of finite etale maps. In the local setting, $X_\infty \to X$ is a covering in the pro-etale site by one open subset. We also have the usual etale sites X_{et} and $\mathfrak{X}_{\mathrm{et}}$.

There is a map of sites $v : X_{\mathrm{proet}} \to X_{\mathrm{et}}$. We have the decompleted structure sheaves $\mathcal{O}_X^+ := v^*\mathcal{O}_{X_{\mathrm{et}}}^+, \mathcal{O}_X := \mathcal{O}_X^+[1/p]$ and the completed structure sheaves $\widehat{\mathcal{O}}_X^+ := \varprojlim \mathcal{O}_X^+/p^n, \widehat{\mathcal{O}}_X := \widehat{\mathcal{O}}_X^+[1/p]$. For example, in the local setting, we have $\mathrm{H}^0_{\mathrm{proet}}(X_\infty, \mathcal{O}_X) = R_\infty = R_\infty^+[1/p]$ and $\mathrm{H}^0_{\mathrm{proet}}(X_\infty, \widehat{\mathcal{O}}_X) = \widehat{R}_\infty$.

Finally, we set $r = v_p(\zeta_p - 1) = \frac{1}{p-1}$.

The p-adic Simpson correspondence gives an equivalence between two categories: one of generalized representations, and one of Higgs modules. In the next sections we shall introduce these two categories; then a period sheaf required to produce functors between them; and finally we shall state the correspondence itself.

2. Generalized representations

2.1. Generalized representations in the local setting.

Definition 2.1. 1. An integral a-small generalized representation is a finite free-$\widehat{R}_\infty^+$ module M_∞^+ of some rank l, endowed with a semilinear Γ_∞ action, such that there is a Γ_∞-equivariant isomorphism $M_\infty^+/p^{a+r} \cong \left(\widehat{R}_\infty^+/p^{a+r}\right)^l$ (in short, it is trivial mod p^{a+r}).

2. An a-small generalized representation is a finite free-$\widehat{R}_\infty$ module M_∞, endowed with a semilinear Γ_∞ action, such that there exists a sublattice $M_\infty^+ \subset M_\infty$ which is an integral a-small generalized representation.

Example 2.1. 1. Let $\delta : \Gamma_\infty \to \widehat{R}_\infty$ be a character such that for a set of generators γ_i of Γ_∞ we have $\delta(\gamma_i) \equiv 1 \bmod p^{a+r}$. Then $M_\infty = \widehat{R}_\infty(\delta)$ is an a-small generalized representation.

2. Take a linear R^+ representation M^+ given by the homomorphism $\Gamma_\infty \to \left(I + p^{(a+r)} M_l(R^+)\right) \cap \mathrm{GL}_l(R^+)$. Then $M_\infty := \widehat{R}_\infty \otimes_{R^+} M^+$ will be an a-small generalized representation. In fact, Wang proves that every a-small generalized representation is of this form (Theorem 4.5).

2.2. Generalized representations in the global setting.

Definition 2.2. 1. An integral a-small generalized representation on is a locally finite free $\widehat{\mathcal{O}}_X^+$-module of rank l satisfying $\mathcal{L}^+/p^{a+r} \cong \left(\widehat{\mathcal{O}}_X^+/p^{a+r}\right)^l$.

2. An a-small generalized representation is locally finite free $\widehat{\mathcal{O}}_X$-module $\mathcal{L}$ of rank l such that locally on X_{proet} it admits a sublattice $\mathcal{L}^+ \subset \mathcal{L}$ which is an integral a-small generalized representation.

Example 2.2. 1. Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $\mathfrak{X}_{\text{ét}}$. Locally on $\mathfrak{X}_{\text{ét}}$, $\mathbb{L}$ admits a $\mathbb{Z}_p$-lattice $\mathbb{L}^+$ such that $\mathbb{L}^+/p^n$ is trivial as a $\mathbb{Z}_p/p^n$ -local system for some fixed $n > a + r$. Then $\mathcal{L}^+ = \mathcal{L}(\mathbb{L})^+ := \widehat{\mathcal{O}}_X^+ \otimes_{\mathbb{Z}_p} \mathbb{L}^+$ is an integral a-small generalized representation and $\mathcal{L} = \mathcal{L}(\mathbb{L}) := \widehat{\mathcal{O}}_X \otimes_{\mathbb{Z}_p} \mathbb{L}$ is an a-small generalized representation.

2. The generalized representations in the global setting do specialize to these in the local setting when $\mathfrak{X} = \mathrm{Spf} R^+$ and it admits local coordinates, in a sense which will now be explained. This is not formal but requires proof, mainly having to do with the behaviour of reductions mod power of p and taking global sections. On the one hand, if $\mathcal{L}$ is an a-small generalized representation in the sense of definition 2.2, then the global sections $(M_\infty, M_\infty^+) = \left(\mathrm{H}^0_{\mathrm{proet}}(X_\infty, \mathcal{L}), \mathrm{H}^0_{\mathrm{proet}}(X_\infty, \mathcal{L}^+)\right)$ is an a-small generalized representation in the sense of definition 2.1 (Lemma 6.14 of Wang). On the other hand, if M_∞ is given as in definition 2.1, there is a unique a-small generalized representation corresponding to it with $(M_\infty, M_\infty^+) = \left(\mathrm{H}^0_{\mathrm{proet}}(X_\infty, \mathcal{L}), \mathrm{H}^0_{\mathrm{proet}}(X_\infty, \mathcal{L}^+)\right)$, because X_∞ is affinoid perfectoid, so the theory of sheaves on affinoid perfectoids behaves like that of quasi-coherent sheaves on affines.

3. There will be more examples arising from Higgs modules via the Simpson correspondence of section 5.

3. Higgs modules

3.1. Higgs bundles in the local setting.

Definition 3.1. 1. An integral a-small Higgs bundle is a pair (H^+, θ_{H^+}) where H is a finite free R^+-module H^+ and θ_{H^+} is an R^+-linear morphism

$$\theta_{H^+} : H^+ \to H^+ \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1)$$

such that $\theta_{H^+}(H^+) \subset p^{(a+r)} H^+ \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1)$ and $\theta_{H^+} \wedge \theta_{H^+} = 0$. This latter condition means that if we consider the natural map

$$s : \widehat{\Omega}^1_{R^+}(-1) \otimes \widehat{\Omega}^1_{R^+}(-1) \to \widehat{\Omega}^2_{R^+}(-2), \omega \otimes \eta \mapsto \omega \wedge \eta$$

then the composition

$$H^+ \xrightarrow{\theta_{H^+}} H^+ \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \xrightarrow{(\mathrm{Id} \otimes s) \circ (\theta_{H^+} \otimes \mathrm{Id})} H^+ \otimes_{R^+} \widehat{\Omega}^2_{R^+}(-2)$$

in equal to 0.
2. An a-small Higgs bundle is a finite free R-module H on R-linear morphism

$$\theta_H : H \to H \otimes_R \widehat{\Omega}^1_R(-1)$$

with $\theta_H \wedge \theta_H = 0$ such that there is an R^+-sublattice H^+ such that $(H^+, \theta_H|_{H^+})$ is an integral a-small Higgs bundle.

Example 3.1. 1. Let $H = R^l$ and let $A_i \in p^{a+r} M_l(R^+)$ for $i = 1, ..., d$. Set

$$\theta_H : H \to H \otimes_R \widehat{\Omega}^1_R(-1)$$
$$\theta_H(x) = \sum_i A_i x \otimes \frac{d \log T_i}{t}.$$

Then (H, θ_H) is a Higgs bundle.
2. There will be more examples arising from generalized representations via the Simpson correspondence of section 5.

3.2. Higgs bundles in the global setting.

The definition in the global setting is a straightforward generalization of the of the local setting.

Definition 3.2. 1. An integral a-small Higgs bundle is a pair $(\mathcal{H}^+, \theta_{\mathcal{H}^+})$ where $\mathcal{H}$ is alocally finite free $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{H}^+$ on $\mathfrak{X}_{\text{ét}}$ and $\theta_{\mathcal{H}^+}$ is an $\mathcal{O}_{\mathfrak{X}}$-linear morphism

$$\theta_{\mathcal{H}^+} : \mathcal{H}^+ \to \mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1)$$

such that $\theta_{\mathcal{H}^+}(\mathcal{H}^+) \subset p^{(a+r)} \mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1)$ and $\theta_{\mathcal{H}^+} \wedge \theta_{\mathcal{H}^+} = 0$. This latter condition means that if we consider the natural map

$$s : \widehat{\Omega}^1_{\mathfrak{X}}(-1) \otimes \widehat{\Omega}^1_{\mathfrak{X}}(-1) \to \widehat{\Omega}^2_{\mathfrak{X}}(-2), \omega \otimes \eta \mapsto \omega \wedge \eta$$

then the composition

$$\mathcal{H}^+ \xrightarrow{\theta_{\mathcal{H}^+}} \mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1) \xrightarrow{(\mathrm{Id} \otimes s) \circ (\theta_{\mathcal{H}^+} \otimes \mathrm{Id})} \mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^2_{\mathfrak{X}}(-2)$$

in equal to 0.

2. An a-small Higgs bundle is a locally finite free $\mathcal{O}_{\mathfrak{X}}[1/p]$-module $\mathcal{H}$ on $\mathfrak{X}_{\text{ét}}$ together with an $\mathcal{O}_{\mathfrak{X}}$-linear morphism

$$\theta_{\mathcal{H}^+} : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1)$$

with $\theta_{\mathcal{H}} \wedge \theta_{\mathcal{H}} = 0$ such that locally on $\mathfrak{X}_{\text{ét}}$ there is an $\mathcal{O}_{\mathfrak{X}}$ sublattice $\mathcal{H}^+$ such that $(\mathcal{H}^+, \theta_{\mathcal{H}}|_{\mathcal{H}^+})$ is an integral a-small Higgs bundle.

4. The period sheaf $\mathcal{O}\mathbb{C}^\dagger$

4.1. **The ring $S^\dagger_\infty$.** Work in the local setting. To construct the ring $S^\dagger_\infty$, one constructs a for each ρ with $|\rho| < \frac{1}{p-1}$ a canonical short exact of $\widehat{R}^+_\infty$-modules, called the Faltings extension. It is of the form

$$0 \to \widehat{R}^+_\infty \to E^+_\rho \to \rho \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \to 0.$$

This makes E^+_ρ into a free $\widehat{R}^+_\infty$-module of rank $d+1$. We then have natural maps

$$\mathrm{Sym}^n E^+_\rho \to \mathrm{Sym}^{n+1} E^+_\rho,$$

$$x_1 \otimes ... \otimes x_n \mapsto 1 \otimes x_1 \otimes ... \otimes x_n.$$

We then obtain the $\widehat{R}^+_\infty$-algebras

$$\widehat{S}^{+,\rho}_\infty = \left(\lim_{n\to\infty} \mathrm{Sym}^n E^+_\rho\right)^\wedge_p$$

$$S^{+,\dagger}_\infty = \lim_{\rho\to 0} S^{+,\rho}_\infty$$

$$S^\dagger_\infty = S^{+,\dagger}_\infty[1/p].$$

We have the following description. Set $Y_i \in E^+_\rho$ for lifts of $1 \otimes \frac{d\log T_i}{t}$. Then

$$\widehat{S}^{+,\rho}_\infty \cong \widehat{R}^+_\infty \langle \rho Y_1, ..., \rho Y_d \rangle$$

which makes $S^\dagger_\infty$ into a limit of rings of functions on smaller and smaller closed discs in the variables $Y_1, ..., Y_d$.

The ring $S^\dagger_\infty$ (as well as the other versions) has two structures apart from the $\widehat{R}^+_\infty$-algebra structure.

1. It has an $\widehat{R}^+_\infty$-linear operator

$$\nabla : S^\dagger_\infty \to S^\dagger_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1)$$

induced from the Faltings extension. It is uniquely determined by the conditions:

(i).∇ is $\widehat{R}^+_\infty$-linear;

(ii). $\nabla(xy) = x\nabla(y) + y\nabla(x)$;

(iii) $\nabla(Y_i) = 1 \otimes \frac{d\log T_i}{t}$;

(iv) ∇ is p-adically continuous.

It is a Higgs field, which means that $\nabla \wedge \nabla = 0$, in a similar sense to the condition appearing in section 3 for θ.

2. The ring $S_\infty^\dagger$ also has an action of Γ_∞ induced from the Faltings extension. It is determined by it being $\widehat{R}_\infty^+$-semilinear and by

$$\gamma_j(Y_i) = Y_i + \delta_{ij}$$

for the element $\gamma_j \in \Gamma_\infty$ which acts by

$$\gamma_j(T_i^{(1/p^m)}) = \zeta_{p^m}^{\delta_{ij}} T_i^{(1/p^m)}.$$

The action of Γ_∞ commutes with that of ∇ (when we let Γ_∞ act trivially on $\widehat{\Omega}^1_{R^+}(-1)$). This follows by direct computation from the identity

$$(\gamma_j \circ \nabla)(Y_i^n) = n(Y_i + \delta_{ij})^{n-1} \otimes \frac{d\log T_i}{t} = (\nabla \circ \gamma_j)(Y_i^n).$$

4.2. **The period sheaf $\mathcal{O}\mathbb{C}^\dagger$.** The sheaf $\mathcal{O}\mathbb{C}^\dagger$ is a sheaf version of the ring $S_\infty^\dagger$. This is essentially straightforward since the Faltings extension sheafifies well. More precisely, the Faltings extension becomes an extension of $\widehat{\mathcal{O}}_X^+$-modules

$$0 \to \widehat{\mathcal{O}}_X^+ \to \mathcal{E}_\rho^+ \to \rho\widehat{\mathcal{O}}_X^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1) \to 0,$$

and by using a similar procedure we obtain a sheaf $\mathcal{O}\mathbb{C}^\dagger$ on X_{proet}. Similar to before, it is an $\widehat{\mathcal{O}}_X$-algebra, endowed with a $\widehat{\mathcal{O}}_X$-linear operator

$$\nabla : \mathcal{O}\mathbb{C}^\dagger \to \mathcal{O}\mathbb{C}^\dagger \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1)$$

which is a Higgs field. It has an action of Γ_∞ when we define these correctly in the sense of actions of groups on sheaves.

If we are in the local situation (so that $\mathfrak{X} = \mathrm{Spf} R^+$ admits toric coordinates as usual), then $\mathrm{H}^0_{\text{proet}}\left(X_\infty, \mathcal{O}\mathbb{C}^\dagger\right) = S_\infty^\dagger$ and all the structure of $\mathcal{O}\mathbb{C}^\dagger$ induces the structure of $S_\infty^\dagger$described in 4.1.

5. The p-adic Simpson correspondence

5.1. **The local p-adic Simpson correspondence.** In this subsection we work in the local setting so that $\mathfrak{X} = \mathrm{Spf} R^+$ and admits local coordinates. The local p-adic Simpson correspondence is a $\otimes$-equivalence of categories

$$\{a-\text{small generalized represntations } M_\infty\} \cong \{a-\text{small Higgs modules } H\}.$$

Let us define the functors in both directions.

There is a functor

$$H : \{a-\text{small generalized representations } M_\infty\} \to \{a-\text{small Higgs modules } H\},$$

$$H(M_\infty) := \left(M_\infty \otimes_{\widehat{R}_\infty} S_\infty^\dagger\right)^{\Gamma_\infty = 1}.$$

One can show that $H(M_\infty)$ is finite free and $\mathrm{rank}_R H(M_\infty) = \mathrm{rank}_{\widehat{R}_\infty} M_\infty$. The Higgs operator $\theta_{H(M_\infty)}$ is defined as follows. There is a map

$$\mathrm{Id}_{M_\infty} \otimes \nabla : M_\infty \otimes_{\widehat{R}_\infty} S_\infty^\dagger \to M_\infty \otimes_{\widehat{R}_\infty} S_\infty^\dagger \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1).$$

Taking Γ_∞ invariants and noting $\mathrm{Id}_{M_\infty} \otimes \nabla$ commutes with Γ_∞ (since ∇ does), we obtain the desired Higgs operator

$$\theta_{H(M_\infty)} : \left(M_\infty \otimes_{\widehat{R}_\infty} S_\infty^\dagger\right)^{\Gamma_\infty} \to \left(M_\infty \otimes_{\widehat{R}_\infty} S_\infty^\dagger\right)^{\Gamma_\infty} \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1).$$

In the converse direction, we have the functor

$$M_\infty : \{a - \text{small Higgs modules } H\} \to \{a - \text{small generalized represntations } M_\infty\},$$

$$M_\infty(\mathrm{H}) = \left(\mathrm{H} \otimes_{R+} S_\infty^\dagger\right)^{\nabla_{\mathrm{H}=0}}$$

where $\nabla_H := \theta_H \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla$. The Γ_∞-action is induced by letting Γ_∞ act trivially on H. The two functors are inverses of each other.

Example 5.1. 1. Suppose that $d = 1$ (for simplicity) and that $M_\infty = M \otimes_R \widehat{R}_\infty$ as in example 2.1.2 (this is always possible). Let γ be a generator of $\Gamma_\infty \cong \mathbb{Z}_p$, and write $A = \mathrm{Mat}(\gamma)$ for the matrix of γ in a basis of M^+. By the a-smallness assumption, $A - I \equiv 0 \bmod p^{a+r}$. Then Wang computes from definition that

$$H(M_\infty) = A^{-Y} M.$$

More precisely,

$$H(M_\infty) = (A^{-1})^Y M = (I + Y(A^{-1} - I) + \binom{Y}{2}(A^{-1} - I)^2 + ...)M \subset M \otimes_{\widehat{R}_\infty} S_\infty^\dagger,$$

which is a finite free R-module of rank equal to $\mathrm{rank}_{\widehat{R}_\infty} M_\infty$.

Indeed, we see it has the correct rank, and we can compute that is fixed by the action of γ, since for $x = A^{-Y}y \in A^{-Y}M$ we have

$$\gamma(x) = \gamma(A^{-Y}y) = \gamma(A^{-Y})Ay = A^{-(Y+1)}Ay = Ay = x.$$

The Higgs operator θ_H is given by $\partial/\partial Y \otimes \frac{d\log T}{t}$. Since $\partial/\partial Y \left(A^{-Y}\right) = (-\log A)\, A^{-Y}$, we see that

$$\mathrm{Mat}(\theta_H) = -\log(A) \otimes \frac{d\log T}{t}$$

if we choose the basis of $H(M_\infty)$ which is A^{-Y} applied to the original basis of M^+.

Conversly, Wangalso computes that if we start from $\left(H, \mathrm{Mat}\left(\theta_H\right) = B \otimes \frac{d\log T}{t}\right)$ then we have $M = \exp(-B)H \subset H \otimes_R S_\infty^\dagger$.

Written briefly, we have a correspondence

$$(M_\infty, \mathrm{Mat}(\gamma) = A) \mapsto \left(H, \mathrm{Mat}\left(\theta_H\right) = -\log(A) \otimes \frac{d\log T}{t}\right),$$

and conversely

$$\left(M_\infty, \mathrm{Mat}(\gamma) = e^{-B}\right) \leftarrow\!\shortmid \left(H, \mathrm{Mat}\left(\theta_H\right) = B \otimes \frac{d\log T}{t}\right)$$

2. As a more specific case in dimension 2, take M_∞ to be the a-small generalized representation with

$$\mathrm{Mat}_{M^+}(\gamma) = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}.$$

Then $\mathrm{Mat}\left(\theta_H\right) = \begin{pmatrix} 0 & -\gamma \\ 0 & 0 \end{pmatrix} \otimes \frac{d \log T}{t}$.

5.2. **The global p-adic Simpson correspondence.** The global setting is similar to the previous subsection, but what requires a little bit of explanation is the description of the functors, which on the face of it will look different.

The global p-adic Simpson correspondence is a $\otimes$-equivalence of categories

$$\{a - \text{small generalized represntations } \mathcal{L} \text{ on } X_{\text{proet}}\} \cong \{a - \text{small Higgs bundles on } \mathfrak{X}_{\text{ét}}\}.$$

Let us define the functors in both directions.

There is a functor

$$\mathcal{H} : \{a - \text{small generalized represntations } \mathcal{L} \text{ on } X_{\text{proet}}\} \to \{a - \text{small Higgs bundles on } \mathfrak{X}_{\text{ét}}\},$$

$$\mathcal{H}(\mathcal{L}) := Rv_* \left(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger\right),$$

with Higgs operator $\theta_{\mathcal{H}(\mathcal{L})} := v_* \left(1 \otimes \nabla\right)$. In fact the complex $Rv_* \left(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger\right)$ turns out to be discrete, so we can just think of the ordinary pushforward $v_* \left(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger\right)$.

In the converse direction, we have the functor

$$\mathcal{L} : \{a - \text{small Higgs bundles on } \mathfrak{X}_{\text{ét}}\} \to \{a - \text{small generalized represntations } \mathcal{L} \text{ on } X_{\text{proet}}\},$$

$$\mathcal{L}(\mathcal{H}) = \left(\mathcal{H} \otimes_{\mathcal{O}_x} \mathcal{O}\mathbb{C}^\dagger\right)^{\nabla_{\mathcal{H}}=0},$$

where $\nabla_{\mathcal{H}} = \theta_{\mathcal{H}} \otimes \mathrm{id} + \mathrm{id}_{\mathcal{H}} \otimes \nabla$. The Γ_∞-action is induced by letting Γ_∞ act trivially on $\mathcal{H}$. The two functors are inverses of each other.

Now let us explain what is the relation between the functor H of the previous subsection and the functor $\mathcal{H}$. The idea is that in the local setting, we have

$$H\left(\mathrm{H}^0_{\text{proet}}\left(X_\infty, \mathcal{L}\right)\right) = \mathrm{H}^0_{\text{ét}}\left(\mathcal{H}\left(\mathcal{L}\right)\right),$$

so in fact the two definitions coincide when we identify $\mathcal{L}$ with $M_\infty = \mathrm{H}^0_{\text{proet}}\left(X_\infty, \mathcal{L}\right)$. The reason for this is Lemma 6.17 of Wang's paper, which relates Galois cohomology with respect to Γ_∞ (as in the functor H) to the pro etale cohomology of $\mathcal{L}$ (as in the functor $\mathcal{H}$, which is defined via the pushforward $v_* \left(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger\right)$, whose sections are ultimately computed in terms of pro etale cohomology).