THE METHOD OF SEN

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Abstract. These are notes for a talk which introduces the method of Sen, a method which allows one to attach to a p-adic Galois representation a canonical linear operator whose eigenvalues are the generalized Hodge-Tate weights of the representation.

1. p-adic Galois representations

Let K be a finite extension of $\mathbb{Q}_p$ and let $G_K = \mathrm{Gal}\left(\overline{K}/K\right)$. By a p-adic Galois representation we shall mean a finite-dimensional continuous representation V of G_K with coefficients in $\mathbb{Q}_p$. These objects are ubiquitous in number theory.

Example 1.1. 1. The cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ is given by the action of elements of G_K on μ_{p^∞}, the roots of unity whose order is a power of p. Letting $K_\infty = K(\mu_{p^\infty})$, it maps $\Gamma = \mathrm{Gal}(K_\infty/K)$ isomorphically onto an open subgroup of $\mathbb{Z}_p^\times$. Its kernel is $H = \mathrm{Gal}(\overline{K}/K_\infty)$, and $G_K/H = \Gamma$.

Example 1.2. 2. Let E be an elliptic curve defined over K. The torsion points $E[p^n](\overline{K})$ for $n \geq 1$ gives rise to an inverse system which is a free $\mathbb{Z}_p$-module of rank 2. It carries a natural action of G_K, and its $\mathbb{Q}_p$ extension is a 2-dimensional p-adic Galois representation.

From now on, fix such a representation V and let d be its dimension. In Sen's method, one studies the *semilinear* representation $W = V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ to obtain information about V. It will turn out that W is much simpler than V, so it will prove fruitful to study W instead.

2. Descent from $\mathbb{C}_p$ to $\widehat{K_\infty}$

To state the main theorem of this section, let us recall the following result.By Galois theory, we know that if $L \subset \overline{\mathbb{Q}}_p$ then $\overline{\mathbb{Q}}_p^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)} = L$. It turns out that an analogous statement holds for $\mathbb{C}_p^{\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/L\right)}$.

Theorem 2.1. *(Ax-Sen-Tate) If $\mathbb{Q}_p \subset L \subset \overline{\mathbb{Q}}_p$ then $\mathbb{C}_p^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)} = \widehat{L}$.*

In particular, we have $\mathbb{C}_p^H = \widehat{K_\infty}$.

Theorem 2.2. *(Sen) We have $\mathbb{C}_p \otimes_{\widehat{K_\infty}} W^H \cong W$. In other words, we can descend W from $\mathbb{C}_p$ to $\widehat{K_\infty}$.*

This is a completed analog of Galois descent. We would like to explain what goes into the proof of this theorem, and for this purpose it is useful to go through the analogous steps of the proof of the usual Galois descent.

First, we recall the definition of nonabelian H^1. Given a group G acting from the left on a group A, we can form the pointed set

$$\mathrm{H}^1(G,A) = \{\xi : G \to A | \xi_{\sigma\tau} = \sigma(\xi_\tau)\xi_\sigma\} / \sim,$$

the equivalence relation being

$$\xi \sim \zeta \text{ if } \exists P \in A \text{ such that } \xi_\sigma = \sigma(P)\zeta_\sigma P^{-1} \text{ for all } \sigma \in G.$$

Given W, we have a 1-cocycle $\xi_W \in \mathrm{H}^1_{\mathrm{cont}}(H, \mathrm{GL}_d(\mathbb{C}_p))$ obtained by choosing a basis of W and letting $\xi_W(h) = \mathrm{Mat}(h)$. This cocycle does not depend on the choice of the basis!

Lemma 2.1. *The following are equivalent.*

1. $\mathbb{C}_p \otimes_{\widehat{K_\infty}} W^H \cong W$.

2. $\xi_W = 1$.

3. ξ_W *is in the image of the inflation map* $\mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(\widehat{K_\infty})) \to \mathrm{H}^1_{\mathrm{cont}}(G_K, \mathrm{GL}_d(\mathbb{C}_p))$.

Proof. If $\mathbb{C}_p \otimes_{\widehat{K_\infty}} W^H \cong W$ then W^H spans a $\mathbb{C}_p$-basis on which the action is trivial, so 1 implies 2. Conversely, to show that 2 implies 1, suppose that $\xi_W = 1$. Then ξ_W is a cobundary, which means that there exists a basis B of W and a matrix $P \in \mathrm{GL}_d(\mathbb{C}_p)$ such that $\mathrm{Mat}_B(h) = h(P^{-1})P$ for any $h \in H$. Changing basis to $B' = P \cdot B$ then gives $\mathrm{Mat}_{B'}(h) = h(P)\mathrm{Mat}_B(h)P^{-1} = 1$ for any $h \in H$, so $B' \subset W^H$ and hence $\mathbb{C}_p \otimes_{\widehat{K_\infty}} W^H \cong W$. Finally, the equivalence between 2 and 3 follows from the inflation-restriction sequence. □

This lemma shows proving Theorem 2.2 is equivalent to proving the following.

Theorem 2.3. $\mathrm{H}^1_{\mathrm{cont}}(H, \mathrm{GL}_d(\mathbb{C}_p)) = 1$.

Proof. (Sketch) The idea is to approximate the proof of Hilbert's theorem 90. In the original proof in the context of a finite Galois extension L/K, one averages the cocycle by taking

$$P = \sum_{\sigma \in \mathrm{Gal}(L/K)} \xi_\sigma \sigma(c)$$

for some $c \in K$. Then $\xi_\sigma = P\sigma(P)^{-1}$ for all $\sigma \in \mathrm{Gal}(L/K)$, so ξ is coboundary. In our context, the group H is not finite and the cohomology is not discontinuous, so we cannot argue in the same way. Rather, one approximates the argument by showing that by carefully choosing elements c_n which are algebraic over K_∞, one can find a sequence of P_n's that converge to an element P with $\xi_\sigma = P\sigma(P)^{-1}$. To choose these c_n's, one uses the theorem of Tate that for any finite extension L of K_∞ one has $\mathrm{Tr}_{L/K_\infty}(\mathcal{O}_L) \supset m_{K_\infty}$. □

Remark 2.1. Scholze has generalized Tate's theorem (in appropriate sense) to any perfectoid field. Therefore, the proof goes through in the same way and the theorem above can be generalized to the statement that $\mathrm{H}^1_{\mathrm{cont}}(\mathrm{Aut}(\mathbb{C}_p/E), \mathrm{GL}_d(\mathbb{C}_p)) = 1$ for any perfectoid field $E \subset \mathbb{C}_p$.

3. Descent from $\widehat{K_\infty}$ to K_∞

The main theorem of this section is the following.

Theorem 3.1. *The map* $\mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(K_\infty)) \to \mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(\widehat{K_\infty}))$ *is an isomorphism.*

This means that W^H further descends to K_∞. Combining the theorem with the results of the previous section, we give an equivalent formulation:

Theorem 3.2. *There exists a unique* K_∞*-vector space* $\mathrm{D}_{\mathrm{Sen}}(V)$ *of* $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ *which is* Γ*-stable, and such that*

$$\mathbb{C}_p \otimes_{K_\infty} \mathrm{D}_{\mathrm{Sen}}(V) \xrightarrow{\sim} \mathbb{C}_p \otimes_{\mathbb{Q}_p} V.$$

Further, $\mathrm{D}_{\mathrm{Sen}}(V)$ *is characterized as the union of all finite dimensional* Γ*-stable subspaces of* $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$*.*

To prove this theorem, our main tools will be the normalized trace maps of Tate. These are maps $R_m : \widehat{K_\infty} \to K_m$ defined by taking the limit over

$$\frac{1}{[K_{n+m} : K_m]} \mathrm{Tr}_{K_{n+m}/K_m} : K_{n+m} \to K_m.$$

The existence of this limit in $\widehat{K_\infty}$ is a nontrivial fact which heavily depends on the ramification theory of $\widehat{K_\infty}$.[1]

We are going to exploit the structure of $\Gamma = \mathrm{Gal}(K_\infty/K)$. Let Γ_m be the subgroup of Γ corresponding to K_m and let γ_m be a topological generator of Γ_m.

Proposition 3.1. *The maps* R_m *satisfy the following properties.*

1. R_m *is a* Γ*-equivariant and* K_m*-linear.*

2. R_m *is a projection onto* K_m *and* $\gamma_m - 1$ *acts invertibely on its kernel.*

3. There exists a constant $c > 0$ *such that for any* $n \geq 1$ *and for any* $x \in \ker R_m$*, one has* $v\left(\left(\gamma_m - 1\right)(x)\right) \geq v(x) - c$*.*

As a first step, we shall use the trace maps to prove the following.

Proposition 3.2. *The map* $\mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(K_\infty)) \to \mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(\widehat{K_\infty}))$ *is injective.*

Proof. Let $\xi \in \mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(K_\infty))$ be a cocycle which is trivial in $\mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(\widehat{K_\infty}))$, and let γ be a topological generator of Γ. There is some m such that $\xi_\gamma \in \mathrm{GL}_d(K_m)$, and the cocycle relation then implies that $\xi_\sigma \in \mathrm{GL}_d(K_m)$ for all $\sigma \in \Gamma$. The assumption means that there exists a matrix $P \in \mathrm{GL}_d(\widehat{K_\infty})$ with $\xi_\sigma = P\sigma^{-1}(P)$. Applying R_m to this equality yields $\xi_\sigma = R_m(P)\sigma^{-1}(R_m(P))$, so ξ is trivial in $\mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(K_\infty))$. □

Proving the surjectivity is more difficult. We do it in several steps.

The following lemma can be interperted as follows. If $A \in \mathrm{GL}_d(\widehat{K_\infty})$ is a matrix sufficiently close to 1, its distance to $\mathrm{GL}_d(K_m)$ can be made smaller by twisted conjugation with a matrix P whose distance to 1 is proportional to the distance between A and $\mathrm{GL}_d(K_m)$.

[1] For example, if we were to consider an extension K_∞/K generated by adjoining all the torsion points of a Lubin-Tate formal group law, such trace maps would usually not exist.

Lemma 3.1. *Let $A \in \mathrm{GL}_d(\widehat{K_\infty})$ and let $T \in \mathrm{GL}_d(K_m)$ such that*

$$v(A-I) \geq 2c,$$

$$v(A-T) \geq rc, \text{ where } r \geq 3.$$

Then there exists $P \in \mathrm{GL}_d\left(\widehat{K_\infty}\right)$ with $v(P-I) \geq (r-1)c$ and $T' \in \mathrm{GL}_d(K_m)$ such that for $A' = B^{-1}A\gamma_m(B)$, we have

$$v(A'-I) \geq 2c,$$

$$v(A'-T') \geq (r+1)c.$$

Proof. Let $T' = R_m(A)$. As $\gamma_m - 1$ acts invertibely on $\ker R_m$, we can write

$$A = T' + (\gamma_m - 1)S,$$

where $T' \in M_d(K_m)$ and $S \in M_d(\ker(R_m))$. Since $v(A-T) \geq rc$, the same is true after applying R_m, so $v(A-T') \geq ac$. Hence

$$v\left(\left(\gamma_m - 1\right)S\right) = v(A-T') \geq rc,$$

so by the properties of R_m, we have $v(S) \geq (r-1)c$.

Let $P = 1 - S$, then $P \in \mathrm{GL}_d\left(\widehat{K_\infty}\right)$ with $v(P-I) \geq (r-1)c$ by construction. Since $v(A-T') \geq rc$ and $v(A-I) \geq 2c$, we also have $v(T'-I) \geq 2c$. It remains to show that for $A' = P^{-1}A\gamma_m(P)$ we have $v(A'-T') \geq (r+1)c$. We compute:

$$A' = P^{-1}A\gamma_m(P) = (1 + S + S^2 + ...)A(1 - \gamma_m(S)).$$

All the terms with S^2 and above have larger valuation than $(r+1)c$, so we get

$$v(A'-T') \geq \min\left\{(r+1)c, v\left((1+S)A(1-\gamma_m(S)) - T'\right)\right\}$$

$$\geq \min\left\{(r+1)c, v\left(A - T' - (\gamma_m(S) - S)\right)\right\} = (r+1)c,$$

where the last equality follows from $A = T' + (\gamma_m - 1)S$. □

We shall also need the following regularization lemma.

Lemma 3.2. *Let $U \in \mathrm{GL}_d(\widehat{K_\infty})$ and let $V_1, V_2 \in \mathrm{GL}_d(K_m)$ such that $v(V_1 - 1), v(V_2 - 1) > c$. Suppose that $\gamma_m(U) = V_1 U V_2$. Then $U \in \mathrm{GL}_d(K_m)$.*

Proof. Let $T = U - R_m(U)$; it suffices to prove that $T = 0$. Since R_m is Γ-equivariant, we have $\gamma_m(T) = V_1 T V_2$. Now notice that

$$\gamma_m(T) - T = V_1 T V_2 - T = \left(V_1 - 1\right) T V_2 + V_1 T \left(V_2 - 1\right) - \left(V_1 - 1\right) T \left(V_2 - 1\right),$$

so that

$$v(T) + c \geq v\left(\left(\gamma_m - 1\right) T\right) \geq v(T) + \inf\left(v(V_1 - 1), v(V_2 - 1)\right) > v(T) + c.$$

But that is impossible unless $v(T) = \infty$, so $T = 0$. □

Proof of Theorem 3.1. It remains to prove the surjectivity of $\mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(K_\infty)) \to \mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(\widehat{K_\infty}))$. Let $\xi \in \mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_d(\widehat{K_\infty}))$ be a cocycle. By continuity, there is an m sufficiently large so that $v\left(\xi_{\gamma_m} - 1\right) \geq 3c$. Starting with $A_1 = \xi_{\gamma_m}$ and $T_1 = 1$, we may apply Lemma 3.1 successively to obtain a sequence of matrices $A_r, P_r \in \mathrm{GL}_d(\widehat{K_\infty})$ and $T_r \in \mathrm{GL}_d(K_m)$, such that

$$A_{r+1} = P_r^{-1} A_r \gamma_m(P_r),$$
$$v(A_r - T_r) \geq (r+2)c,$$
$$v(P_r - I) \geq (r+1)c.$$

Let $P = \prod_{r=1}^{\infty} P_r$. The congruence conditions on each P_r guarantee the convergence of the product. On the other hand, the sequences A_r and T_r converges to the same limit U. As $T_r \in \mathrm{GL}_d(K_m)$ for all r, we have $U \in \mathrm{GL}_d(K_m)$. Applying the relation $A_{r+1} = P_r^{-1} A_r \gamma_m(P_r)$ successively and taking the limit as $r \to \infty$ we obtain the relation $P\xi_{\gamma_m}\gamma_m(P^{-1}) = U$.
Now let ζ be the cocycle defined by $\zeta_\sigma = P\xi_\sigma\gamma_m(P^{-1})$, so that ξ and ζ are cohomologous. The previous computation shows at least for $\sigma = \gamma_m$ we have $\zeta_\sigma \in \mathrm{GL}_d\left(K_m\right)$, and it remains to show this actually holds for all $\sigma \in \Gamma$. To prove this, we write down the cocycle relation

$$\gamma_m(\zeta_\sigma) = \zeta_{\gamma_m}^{-1} \zeta_\sigma \sigma(\zeta_{\gamma_m}).$$

As $v(\xi_{\gamma_m} - 1) \geq 3c > c$, Lemma 3.2 applies with $U = \zeta_\sigma$, $V_1 = \zeta_{\gamma_m}^{-1}$ and $V_2 = \sigma(\zeta_{\gamma_m})$ and we conclude that $\zeta_\sigma \in \mathrm{GL}_d(K_m)$, as required. □

4. The operator Θ_{Sen}

The K_∞-vector space $\mathrm{D}_{\mathrm{Sen}}(V)$ is endowed a Γ-semilinear action. We will now show that this action can be differentiated to obtain a K_∞-linear endomorphism.

Definition 4.1. The operator Θ_{Sen} is the operator of $\mathrm{D}_{\mathrm{Sen}}(V)$ given by the formula

$$\Theta_{\mathrm{Sen}}(x) = \frac{\log(\gamma)}{\log_p \chi(\gamma)}(x) = \frac{1}{\log_p \chi(\gamma)} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\gamma - 1)^n(x)}{n}.$$

The formula is well defined and independent of $\gamma \in \Gamma$ whenever γ is sufficiently close to 1.

Lemma 4.1. *Θ_{Sen} is K_∞-linear.*

Proof. Fix γ sufficiently small so that Θ_{Sen} is defined . The formula makes it clear that Θ_{Sen} is a derivation over the derivation $\Theta = \frac{\log(\gamma)}{\log_p \chi(\gamma)}$ of K_∞, so that for $a \in K_\infty$ and $x \in \mathrm{D}_{\mathrm{Sen}}(V)$, we have

$$\Theta_{\mathrm{Sen}}(ax) = \Theta(a)x + a\Theta_{\mathrm{Sen}}(x).$$

It thus suffices to explain why $\Theta = 0$. Indeed, any $a \in K_\infty$ is fixed by an open subgroup of Γ and hence by γ^m for sufficiently large m. Since $\Theta = \frac{\log(\gamma^m)}{\log_p \chi(\gamma^m)}$ and $\log(\gamma^m)(a) = 0$, we see that $\Theta(a) = 0$. □

Using the Sen operator, we can define the following useful invariants.

Definition 4.2. 1. The generalized Hodge-Tate weights of V are the eigenvalues Θ_{Sen}.
2. If Θ_{Sen} is semisimple with integer eigenvalues, we call these elements the Hodge-Tate weights and we say that V is Hodge-Tate.

Example 4.1. 1. Take $V = \chi^n$ for $n \in \mathbb{Z}$. Clearly Θ_{Sen} is multiplication by n, so V is Hodge-Tate with weight n.

2. V is Hodge-Tate if and only if one has a decomposition

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \oplus_i \mathbb{C}_p(\chi^{n_i}).$$

The sufficiency of the condition is obvious by 1. To show necessity, suppose V is Hodge-Tate. The action of Θ_{Sen} on $\mathrm{D}_{\mathrm{Sen}}(V)$ is semisimple, and integrating Θ_{Sen} determines the action of Γ on $\mathrm{D}_{\mathrm{Sen}}(V)$ up to a finite image character. Decomposing $\mathbb{C}_p \otimes_{K_\infty} \mathrm{D}_{\mathrm{Sen}}(V)$ according to the eigenspaces of Θ_{Sen}, we see that

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \mathbb{C}_p \otimes_{K_\infty} \mathrm{D}_{\mathrm{Sen}}(V) \cong \oplus_i \mathbb{C}_p(\eta_i \chi^{n_i}),$$

where each η_i is a finite image character of G_K. We reduce to proving that $\mathbb{C}_p(\eta) = \mathbb{C}_p$, for which it suffices to show that $\mathrm{H}^0(G_K, \mathbb{C}_p(\eta)) \neq 0$. Taking a finite Galois extension L/K such that $\eta|_{G_L} = 1$, we have

$$\mathrm{H}^0(G_K, \mathbb{C}_p(\eta)) = \mathrm{H}^0\left(\mathrm{Gal}(L/K), \mathrm{H}^0(G_L, \mathbb{C}_p(\eta))\right) = \mathrm{H}^0\left(\mathrm{Gal}(L/K), L(\eta)\right),$$

so we reduce to proving that $\mathrm{H}^0\left(\mathrm{Gal}(L/K), L(\eta)\right) \neq 0$. But this follows from Hilbert's theorem 90.

3. There are representations which are not Hodge-Tate. For example, let V be the two dimensional representation given by

$$g \mapsto \begin{pmatrix} 1 & \log_p \chi(g) \\ 0 & 1 \end{pmatrix},$$

then

$$\Theta_{\mathrm{Sen}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which is not semisimple.

As an illustration of the Galois-theoretic information underlying Θ_{Sen}, we conclude by mentioning the following useful theorem of Sen.

Theorem 4.1. *Let V be a p-adic Galois representation. The following are equivalent.*

1. V is potentially unramified.

2. $\Theta_{\mathrm{Sen}} = 0$.

3. V is Hodge-Tate and all of its Hodge-Tate weights are zero.