THE SPRAGUE-GRUNDY THEOREM

GAL PORAT

Abstract. These are notes for a talk introducing the Sprague-Grundy theorem.

1. Impartial games

Consider the following games, played between two players, Alice and Bob. Alice plays first. A player who cannot move loses.

Example 1.1. There is a single heap of $n \geq 0$ stones. At each player's turn, they take any number of stones from the heap. A player who cannot take any stones loses.

Clearly, this is a stupid game, because if $n > 0$ Alice can just take all the stones on the first turn and win, while Bob wins if $n = 0$. This game has the special name $*n$.

Example 1.2. There is a single heap of n stones. At each player's turn, they take any number of stones between 1 and 7 (say) from the heap.

If the number of stones is between 1 and 7, clearly Alice wins. However, if the number of stones is 8, Bob wins. A moment's thought shows that actually, the same is true modulo 8, so that Alice wins exactly if the number of stones is different from 0 mod 8.

Example 1.3. Two players start with a strip of n white squares and they take alternate turns. On each turn, a player picks two contiguous white squares and paints them black. A player who cannot make a move loses.

Here, it is more complicated to analyze who wins (we'll explain how to do this later). It turns out that Bob wins if $n = 5, 9, 21, 25 \text{ or } 29 \mod 34$, or is equal to one of $1, 15, 35$, and otherwise Alice wins.

All of these are examples of impartial games. An *impartial game* is a game in which the allowable moves depend only on the position and not on which of the two players is currently moving. For example, chess is not an impartial game, because the different players are not allowed to move the same pieces.

If G is a game, we like to write $G = \{G_1, ..., G_m\}$ where G_i are the possible positions for G after making one move. For example, $*n = \{*0, *1, ..., *(n-1)\}$.

2. Equivalence and addition

To analyze these sort of games, it will be useful to introduce some language and operations.

Given a game G, let's say it's of N-type if the next player wins (the one whos turn it is), and it's of P-type if the previous player wins. All of our three previous examples are of N-type.

Games can be added: given two games G and H, we have the game $G+H$, where each player can play their turn at either G or H, until there are no legal moves.

Example 2.1. If G is any game, then $G+G$ is of P-type. Indeed, for any move the first player does, the second player can do the same move in the other game.

Example 2.2. The game $*n + *m$ is of P-type if and only if $n = m$. If $n = m$, this is a special case of the previous example. On the other hand, if, say, $n > m$, the first player can take $n-m$ stones from the first heap, and the second player now has to play $*m + *m$, which is a losing position for them.

Lemma 2.1. *If A is of P-type, then G and $G + A$ have the same type.*

Proof. If G is of P-type, the second player has a winning strategy: respond both in G and in A according to their winning strategy.

On the other hand, if G is of N-type, the first player has a winning strategy for $G + A$. Indeed, as the first player, you can make a move in G, after which the remaining game G' is of P-type. Then the total remaining game is $G' + A$; since both of G' and A are of P-type, this means that $G' + A$ is also of P-type (by the direction we have already shown), so $G + A$ is of N-type. □

We are interested in saying when two games are "equivalent". The relation "having the same type" is a first candidate, but it is too coarse. For example, for each $n, m > 0$, the games $*n$ and $*m$ have the same type, but $*n + *m$ and $*m + *m$ do not have the same type unless $m = n$. For this reason, we give the following definition.

Definition 2.1. We say two games G, G' are equivalent and write $G \approx G'$ if for any game H, the game $G + H$ and $G' + H$ have the same type.

Example 2.3. As explained above, $*n \approx *m$ if and only if $n = m$.

Example 2.4. If A is of P-type, then $G + A \approx G$ for any G indeed. Indeed, if H is any other game, we have that $G + A + H = (G + H) + A$ and $G + H$ have the same type because of Lemma 2.1.

Here is a criterion which determines when two games are equivalent.

Lemma 2.2. *If G and G' are two games, then $G \approx G'$ if and only if $G + G'$ is of type P.*

Proof. If $G \approx G'$ then $G + G'$ has the same type as $G + G$, and we have already seens this is of P-type. Conversely, suppose $G + G'$ is of type P, and let H be any game. We want to show $G + H$ and $G' + H$ have the same type. Well, this is true, because

$$G + H \approx G + H + (G' + G') = (G + G') + G' + H \approx G' + H.$$

□

3. The Sprague-Grundy theorem

The following theorem is due to Sprague and Grundy.

Theorem 3.1. *Any game is equivalent to a unique game of the form* $*n$.

In fact, the theorem also tells you how to find this number, in principle, using the mex operator. This is defined by

$$\text{mex}\,(n_1, ..., n_k) = \text{the minimal nonnegative number not in } \{n_1, ..., n_k\}.$$

The theorem follows from the following Proposition by induction.

Proposition 3.1. *Suppose that in a game G there are k possible moves the first player can do. Suppose moreover they are all equivalent some $*n_i$, so after 1 move, the possibile positions are $*n_1, ..., *n_k$. Then $G \approx *\text{mex}\,(n_1, ...n_k)$.*

Proof. Write $m = \text{mex}\,(n_1, ...n_k)$; to show $G \approx *m$, let's show by Lemma 2.2 that $G + *m$ is of type P. To show this, we find a winning strategy for the second player.

If the next player moves $*m$ to $*m'$ for $m' < m$, then, because m is the minimal excluded number from $n_1, ..., n_k$, the second player can move in G to $*m'$, because m' is not excluded. Then the remaining game is $*m' + *m'$, which the second player wins.

On the other hand, if the next player moves in G to some n_i, then if $n_i < m$, the second player can move in the second game to $*n_i$, so the remaining game is $*n_i + *n_i$, which the second player wins. Otherwise, if $n_i > m$ the second player can then move to m, so the remaining game is $*m + *m$, which the second player wins. □

4. Grundy numbers

The number n in Theorem 3.1 such that $G \approx *n$ is called the *Grundy number* $\text{Gr}(G)$ of G. How do we actually compute this number? If we are interested in whether or not there is a winning strategy, that's all we need to know. Indeed, a game G is of N-type if and only if $\text{Gr}(G) > 0$, because $G \approx *\text{Gr}(G)$.

Proposition 3.1 gives the following:

Proposition 4.1. *If $G = \{*n_1, ..., *n_k\}$, then* $\text{Gr}\,(G) = \text{mex}\,(n_1, .., n_k)$.

This shows that we can compute $\text{Gr}(G)$ recursively. But in practice this is often really difficult because the amount of computation rises exponentially.

One thing we do have a nice answer for is the Grundy number of $G + G'$.

Theorem 4.1. $\text{Gr}\,(*n + *m) = n \oplus m$, *where $\oplus$ means the bitwise XOR of n and m.*

Proof. This is a pleasant exercise by induction, left for the reader! □

Example 4.1. Consider a game between two people, where the are three heaps of stones of heights n_1, n_2 and n_3. Would you prefer to play first or second? How does this depend on n_1, n_2 and n_3?

Another occasion where we can compute Grundy numbers is when some periodicity occurs.

Example 4.2. Consider again example 1.2 from section 1, and call $G(n)$ the game where you play with a heap of size n, where each player takes away each turn a number of stones between 1 to 7. Then I claim that $\mathrm{Gr}(G(n+8)) = \mathrm{Gr}(G(n))$. This can be proven by induction. Suppose it has been shown by direct computation for $n \leq 16$, say, and shown up to $n = m$. To show it holds for $n = m+1$, simply notice that

$$G(m+1) = \{G(m), ..., G(m-7)\} \approx \{G(m-7), ..., G(m-15)\} = G(m-7),$$

so $\mathrm{Gr}(G(n+7)) = \mathrm{Gr}\,(G(n))$. This leads to the formula $\mathrm{Gr}\,(G(n)) = n \mod 8$

Example 4.3. Consider again example 1.3 from section 1, and call $G(n)$ the game played with a strip of length n. One can compute by hand (or in actuality, by computer) that at some point the values $\mathrm{Gr}\,(G(n))$ become periodic with period 34, say, for $70 \leq n \leq 200$. Then one can prove they stay periodic, i.e. $\mathrm{Gr}\,(G(n+34)) = \mathrm{Gr}\,(G(n))$ for all $n \geq 70$. The point is that we can again argue by induction, though this is slightly more delicate than the previous example. Say we know the periodicity for all $m - 68 \leq n < m$. The point is that, no matter what move we make for $G(m)$, we are going to be left with two smaller strips of lengths m_1, m_2 for which we know $G(m_1) \approx G(m_1 - 34)$ and $G\,(m_2) \approx G(m_2 - 34)$. So if we make this move, we can also find a move in $G(m-34)$ which leaves us with the same option. The converse reasoing is also true. This means that there's going to be an equality of sets

$$\{\mathrm{Gr}\,(G_i) : G_i \text{ possible move for } G(m)\}$$

and

$$\{\mathrm{Gr}\,(G_i) : G_i \text{ possible move for } G(m-34)\}$$

(unlike the previous example, the set of moves for $G(m)$ is going to be much larger than the set of moves for $G(m-34)$, but the set of Grundy numbers of these moves is going to be the same!). Taking the mex of both of these sets, we find out that $\mathrm{Gr}\,(G(m)) = \mathrm{Gr}\,(G(m-34))$, and by examining what the period is, we find out that $\mathrm{Gr}(G(n)) = 0$ exactly if $n = 5, 9, 21, 25$ or $29 \mod 34$ or if $n = 1, 15, 35$ (these exceptions happen before the function gets into a period).

Email address: galp@uchicago.edu