STRUCTURE OF THE HECKE ALGEBRA FOR GL_2

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Abstract. In this note I shall try to give a self contained treatment of the basic properties of the (anemic) Hecke algebra for GL_2, and the various interpertations of its ideals. We shall also attempt to visualize these objects.

Often in the context of Galois representations, one wishes to prove some relation between Galois representations with certain properties and representations coming from modular forms. This usually takes up the form of some $R = T$ theorem, which exhibits an equality between some universal deformation ring R and some Hecke algebra T.

The purpose of this note is to prove some central properties of T in the most simple nontrivial case, that of GL_2. We warn the reader that we only consider the anemic Hecke algebra (the Hecke operators away from the level), and thus our terminology can differ from that of authors who work in other contexts.

1. The structure of $\mathbb{T}_k$ over $\mathbb{Z}$

Let $k \geq 1$ be an integer and let $N \geq 1$ be a natural number. We let $\mathcal{M}_k(N) = \mathcal{M}_k(N, \mathbb{Z})$ be the $\mathbb{Z}$-module of modular forms of weight k, level N and coefficients in $\mathbb{Z}$. For $l \nmid N$ we have the Hecke operators T_l and $lS_l = \langle l \rangle\, l^{k-1}$ acting on $\mathcal{M}_k(N)$, and we let $\mathbb{T}_k = \mathbb{T}_k(N)$ be the image of these operators in $\mathrm{End}(\mathcal{M}_k(N))$.

Proposition 1.1. *$\mathbb{T}_k$ is finite flat over $\mathbb{Z}$.*

Proof. This is clear because there is a containment $\mathbb{T}_k \subset \mathrm{End}(\mathcal{M}_k(N))$. □

Therefore, the structure map $\mathrm{Spec}\mathbb{T}_k \to \mathrm{Spec}\mathbb{Z}$ has finite fibers, and satisfies both the going up and going down theorems. This means that there are only two types of prime ideals in $\mathbb{T}_k$:

1. Minimal primes β which lie above the ideal (0), whose residue rings $\mathbb{T}_k/\beta = \mathcal{O}_\beta$ are orders in a number field.

2. Maximal primes m which lie above the ideal (p), whose residue fields are finite extensions of $\mathbb{F}_p$.

Moreover, there are finitely many primes of type 1, and for each p there are finitely many primes of type 2.

Let us discuss in a bit more detail the minimal primes of $\mathbb{T}_k$. By the structure theorem above, we have $\{\text{minimal primes of } \mathbb{T}_k\} \longleftrightarrow \{\text{kernels of homomorphisms } \lambda : \mathbb{T}_k \to \mathbb{C}\}$.We call a homomorphism $\lambda : \mathbb{T}_k \to A$ a system of Hecke eigenvalues, valued in A. By the multiplicity one theorem, we have that the kernel of $\lambda : \mathbb{T}_k \to \mathbb{C}$ determines λ uniquely.

Similar comments hold for morphisms $\lambda : \mathbb{T}_k \to \overline{\mathbb{F}}_p$. Thus we can summarize this discussion in the following proposition.

Proposition 1.2. We have correspondences

$$\{\text{minimal primes of } \mathbb{T}_k\} \longleftrightarrow \{\text{maximal ideals of } \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Q}\} \longleftrightarrow$$

$$\{\text{systems of Hecke eigenvalues } \lambda : \mathbb{T}_k \to \mathbb{C}\},$$

and

$$\{\text{maximal primes of } \mathbb{T}_k\} \longleftrightarrow \bigcup_p \{\text{maximal ideals of } \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{F}_p\} \longleftrightarrow$$

$$\bigcup_p \{\text{systems of Hecke eigenvalues } \lambda : \mathbb{T}_k \to \overline{\mathbb{F}}_p\}.$$

Corollary 1.1. $\mathbb{T}_k$ *is reduced.*

Since $\mathbb{T}_k$ is flat, an equivalent yet more geometric formulation of this statement would be to say that $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Q}$ is etale over $\mathbb{Q}$, or that the generic fiber of $\mathrm{Spec}\mathbb{T}_k$ is etale over $\mathrm{Spec}\mathbb{Q}$.

Proof. We have $\mathbb{T}_k \hookrightarrow \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{C}$, because $\mathbb{T}_k$ is flat. Moreover, the homomorphism

$$(\lambda_f)_f : \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{C} \to \prod_f \mathbb{C}$$

is an isomorphism, by choosing a simultatenous basis of eigenforms $\{f\}$ of $\mathcal{M}_k(N, \mathbb{C})$ for the commuting, normal operators T_l and lS_l for $l \nmid N$. □

Remark 1.1. Notice that in this proof, systems of Hecke eigenvalues could appear twice in the map we have defined: we are *not* saying the basis of eigenforms is parametrized by minimal primes. Only newforms are parametrized by minimal primes. Also note that this corollary and its proof are very much false in general for the full Hecke algebra.

Now let $\{\beta\}$ be the set of minimal primes of $\mathbb{T}_k$. Since $\mathbb{T}_k$ is reduced, the intersection $\cap_\beta \beta$ is 0, and so we have an embedding

$$\mathbb{T}_k \hookrightarrow \prod_\beta \mathbb{T}_k/\beta = \prod_\beta \mathcal{O}_\beta.$$

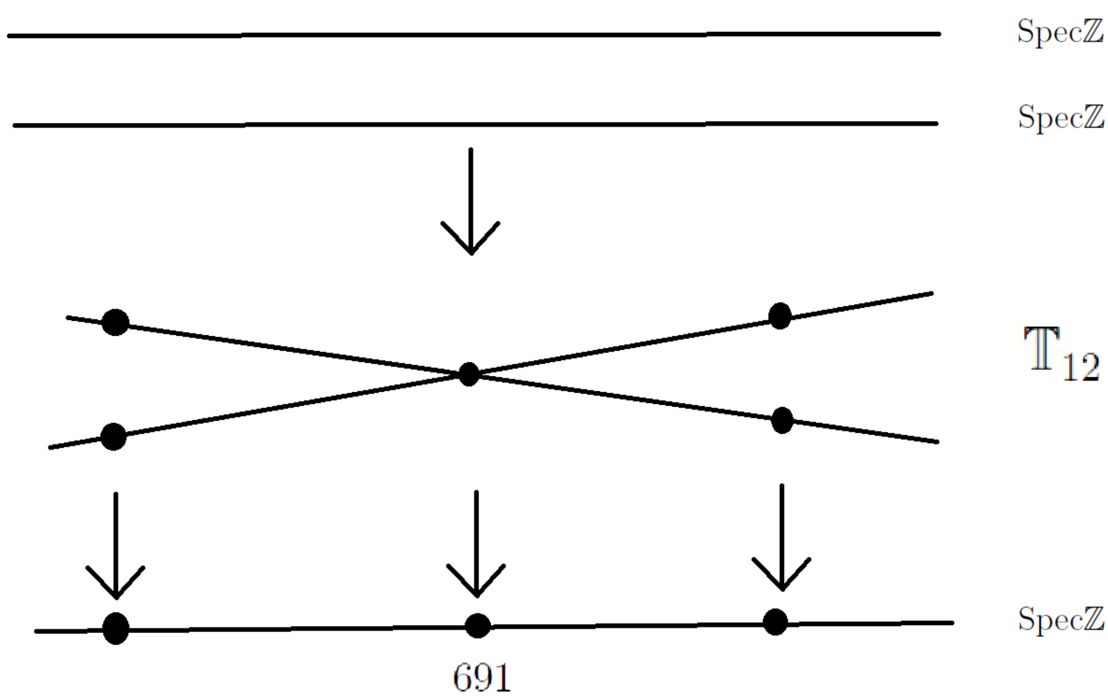
This induces a covering $\amalg_\beta \mathrm{Spec}(\mathcal{O}_\beta) \to \mathbb{T}_k$. Note however this is usually not even a flat covering. Indeed, since both of these schemes are noetherian, if this map was flat it would also be open (by Chevalley's theorem + "closed under generization property" of flat maps + constructible sets closed under generization are open), and this is false if two components intersect in $\mathrm{Spec}\mathbb{T}_k$, as will happen in the example we will give shortly.

In general, these components could have singularities at certain points because orders are not always integrally closed. This will amount to the local ring at the maximal point not being a discrete valuation ring. However, the components themselves cannot intersect in a non transverse way in the generic fiber because $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Q}$ is etale.

Example 1.1. Let's take $N = 1$ and $k = 12$. In this case we have two minimal primes, corresponding to the systems of Hecke eigenvalues $\lambda_\Delta : \mathbb{T}_k \to \mathbb{Z}$ given by the cusp form Δ and $\lambda_{E_{12}} : \mathbb{T}_k \to \mathbb{Z}$ given by the Eisenstein form E_{12}. These give an injection

$$\mathbb{T}_{12} \xrightarrow{\sim} \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \mod 691\} \hookrightarrow \mathbb{Z} \times \mathbb{Z}.$$

Thus $\mathbb{T}_{12}$ has a faithful flat covering by two copies of Spec$\mathbb{Z}$, and these copies in $\mathbb{T}_{12}$ intersect at the fiber over the point $(691) \in$ Spec$\mathbb{Z}$. The fiber of Spec$\mathbb{T}_{12}$ over 691 consists of a unique point, because $\mathbb{T}_{12} \otimes \mathbb{Z}/691 \cong \mathbb{Z}/691$. On the other hand the local ring at this point has two distinct minimal primes corresponding to the components of Δ and E_{12}. Below is the picture we would like to keep in mind for this case.



2. The structure $\mathbb{T}_k$ over $\mathbb{Z}_p$

In this section we consider the algebra $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Geometrically, one can think of this as the fiber of Spec$\mathbb{T}_k$ over Spf$\mathbb{Z}_p \to$ Spec$\mathbb{Z}$. We already see in the example of the previous section that the behaviour of this algebra can heavily depend on p.

Most of the discussion of the previous section carries over. Namely, we have the following.

Lemma 2.1. $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a reduced algebra and it is finite free over $\mathbb{Z}_p$. The algebra $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ has finitely many prime ideals. Moreover, we have the following correspondence:

$$\{\text{minimal primes of } \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p\} \longleftrightarrow \{\text{maximal ideals of } \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Q}_p\} \longleftrightarrow$$
$$\left\{\text{systems of Hecke eigenvalues } \lambda : \mathbb{T}_k \to \overline{\mathbb{Z}}_p\right\},$$

and

$$\{\text{maximal primes of } \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p\} \longleftrightarrow \{\text{maximal ideals of } \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{F}_p\} \longleftrightarrow$$

$$\left\{\text{systems of Hecke eigenvalues } \lambda : \mathbb{T}_k \to \overline{\mathbb{F}}_p\right\}.$$

Proof. Clearly $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is finite flat over $\mathbb{Z}_p$ (and hence, finite free) as $\mathbb{T}_k$ is finite flat over $\mathbb{Z}$. Choosing an isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$, we see that $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p \hookrightarrow \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n$, so $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is reduced. The same logic as before applies to the analysis of prime ideals. □

Most of the remainder of this section will be dedicated to the proof of the following theorem.

Theorem 2.1. $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_m (\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p)_m$, *where the product ranges over the maximal ideals* m.

Given a ring A, let $\mathrm{rad}(A)$ denote its Jacobson radical.

Lemma 2.2. *Let A be a $\mathbb{Z}_p$-algebra such that $A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is Artinian, and such that $p \in \mathrm{rad}(A)$. Then $\mathrm{rad}(A)^n \subset (p) \subset \mathrm{rad}(A)$ for some n. In particular, the completion with respect to (p) coincides with the completion with respect to $\mathrm{rad}(A)$.*

Proof. Since $A/pA = A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is Artinian, its Jacobson radical is nilpotent ([AM], Corollary 8.2 and Proposition 8.4). But on the other hand, $\mathrm{rad}(A/pA) = \mathrm{rad}(A)/p$. □

More generally, let A be a finite free $\mathbb{Z}_p$ algebra. We will prove that $A \cong \prod_m A_m$.

Proof of Theorem 2.1. Since A is finite free over $\mathbb{Z}_p$, it is p-adically complete. All maximal ideals of A have characteristic p, so $p \in \mathrm{rad}(A)$. On the other hand, the ring $A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is a finite $\mathbb{F}_p$-algebra, so it is Artinian. By the lemma, we conclude that A is complete with respect to its radical. On the other hand, A has only finitely many maximal ideals, and they are mutually coprime, so

$$\mathrm{rad}(A) = \cap_m m = \prod_m m,$$

the (finite) product ranging over the maximal ideals of A.

Thus, we have

$$A \cong \varprojlim A/\mathrm{rad}(A)^n \cong \prod_m \varprojlim A/m^n$$
$$= \prod_m \varprojlim (A_m/m)^n = \prod_m \widehat{A_m},$$

where each $\widehat{A_m}$ is the completion of A_m with respect to its m-adic topology. It remains to explain why each A_m is complete with respect to its m-adic topology.

To do so, we note that $\mathrm{rad}(A_m) = mA_m$, and that $p \in A_m$. Moreover, $A_m \otimes_{\mathbb{Z}_p} \mathbb{F}_p = \left(A \otimes_{\mathbb{Z}_p} \mathbb{F}_p\right)_m$ is a localization of an Artinian algebra, hence is Artinian. So by use of Lemma 2.2 again, we conclude that the m-adic completion of A_m conicides with its p-adic completion. We further reduce to proving that A_m is p-adically complete.

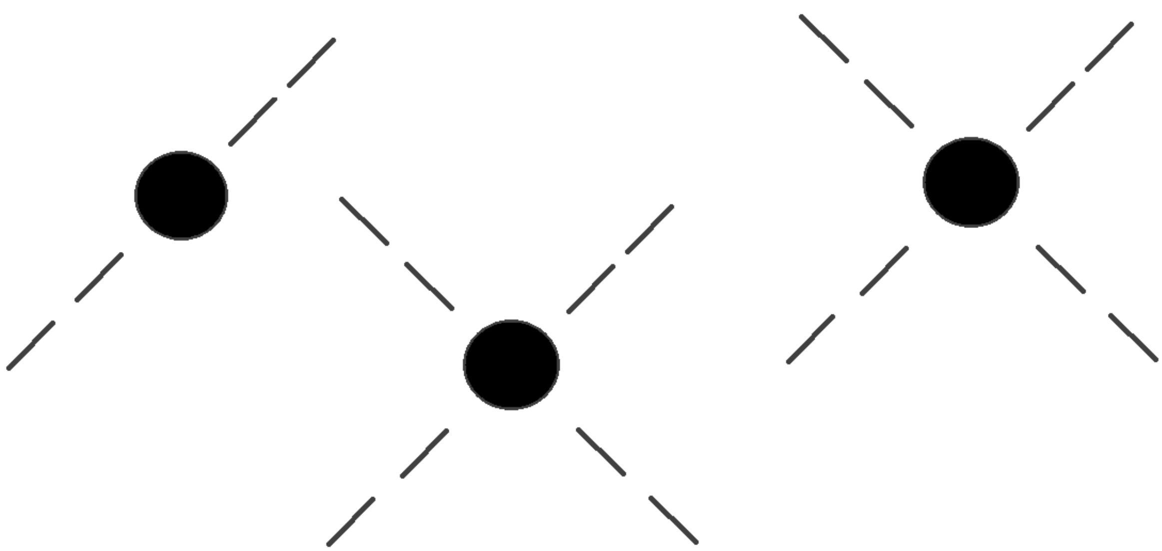
Since we already know that $\widehat{A_m}$ is a quotient of A, we also know it is a finitely generated $\mathbb{Z}_p$-module. As $\mathbb{Z}_p$ is Noetherian, any $\mathbb{Z}_p$-submodule of $\widehat{A_m}$ is also finitely generated over $\mathbb{Z}_p$, and hence is p-adically complete. We conclude by noting that the map $A_m \to \widehat{A_m}$ is injective by Krull's intersection theorem. □

Corollary 2.1. *Every prime ideal of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is contained in a unique maximal ideal of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$.*

Putting all this together, we can visualize $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ as follows. It has finitely many maximal ideals whose residue fields are finite extensions of $\mathbb{F}_p$. For each of these there are local components, corresponding to the minimal primes of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ which are contained in that maximal ideal. Moreover, no tangencies of these local components can occur because $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is reduced.

This is illustrated in the following picture, which is an example of an algebra $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for which there are 3 maximal ideals, two of which have two minimal primes contained in them and one which has one minimal primes contained in it. The closed points, marked by black dots, correspond to maximal ideals, which in turn correspond to $\overline{\mathbb{F}}_p$-valued systems of Hecke eigenvalues. The local components, marked by dotted lines, correspond to minimal primes, which in turn correspond to $\overline{\mathbb{Z}}_p$-valued systems of Hecke eigenvalues. A dotted line goes through a black dot if its corresponding minimal prime is contained the corresponding maximal prime, or what amount to the same, the reduction of its $\overline{\mathbb{Z}}_p$-valued system of Hecke eigenvalues is equal to the $\overline{\mathbb{F}}_p$-valued system of Hecke eigenvalues.

$\mathrm{Spec}\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$



3. The big p-adic Hecke algebra $\mathbb{T}$

Let $\mathbb{T}_k^{(p)}$ denote the sub $\mathbb{Z}$-algebra of $\mathbb{T}_k$ generated by T_l and lS_l for $l \nmid Np$. By the same analysis as before, it shares with $\mathbb{T}_k$ all the properties we have proven in section 1, so it is finite flat over $\mathbb{Z}$, reduced, and so on. Moreover, since systems of Hecke eigenvalues $\lambda : \mathbb{T}_k \to \mathbb{C}$ are already determined by their value on T_l and lS_l for all but finitely many l, the systems of Hecke eigenvalues of $\mathbb{T}_k^{(p)}$ are the same as those of $\mathbb{T}_k$. In fact this applies to show that $\mathbb{T}_k^{(p)}$

has finite index in $\mathbb{T}_k$. Similarly, we have the sub $\mathbb{Z}_p$-algebra $\mathbb{T}_k^{(p)} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and the relation of $\mathbb{T}_k^{(p)} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is analogous to the relation of $\mathbb{T}_k^{(p)}$ to $\mathbb{T}_k$.

From now on, in order to simplify the notation, we shall write $\mathbb{T}_k$ when we really mean $\mathbb{T}_k^{(p)}$. As we shall only use properties of $\mathbb{T}_k$ proved in sections 1 and 2, and as all of these still apply, this will make no difference for us. The real significance of this replacement is for Theorem 3.1 below.

Now let $\mathbb{T}_{\leq k} \hookrightarrow \prod_{i=1}^k \operatorname{End}(\mathcal{M}_i(N))$ be the algebra generated by the diagonal action of the T_l and lS_l. This is of course embedded in (but very much not equal to) $\prod_{i=1}^k \mathbb{T}_i$. We let $\mathbb{T} = \varprojlim \mathbb{T}_{\leq k} \otimes_{\mathbb{Z}} \mathbb{Z}_p$; this is called the p-adic Hecke algebra.

From its definition, we see immediately that $\mathbb{T}$ is reduced, p-torsionfree and p-adically complete. We may think of a continuous homomorphism $\lambda : \mathbb{T} \to \overline{\mathbb{Z}}_p$ as giving a $\overline{\mathbb{Z}}_p$-valued system of Hecke eigenvalues. This generalizes the previous definition in the following sense. For each k, we have a projection $\mathbb{T} \twoheadrightarrow \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and composing it with a system of Hecke eigenvalues $\lambda : \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \overline{\mathbb{Z}}_p$ gives $\lambda : \mathbb{T} \to \overline{\mathbb{Z}}_p$.

Next, we shall want to understand the prime ideals of $\mathbb{T}$. What is the relation of prime ideals of $\mathbb{T}$ to prime ideals of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$?

Well, the prime ideals of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ pullback to these prime ideals of $\mathbb{T}$ which contain the kernel of $\mathbb{T} \to \mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$. In particular, the maximal ideals of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ pullback to maximal ideals of $\mathbb{T}$. On the other hand, minimal ideals of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ very much do not pullback to minimal ideals of $\mathbb{T}$, but rather to coheight 1 ideals. As we will see, $\mathbb{T}$ is no longer finite free over $\mathbb{Z}_p$, so it will no longer be true that $\overline{\mathbb{Z}}_p$-valued systems of Hecke eigenvalues of $\mathbb{T}$ are the same as minimal ideals of $\mathbb{T}$.

On the other hand, recall we had the correspondence between $\overline{\mathbb{Z}}_p$-valued systems of Hecke eigenvalues of $\mathbb{T}_k$ and maximal ideals of $\mathbb{T}_k \otimes_{\mathbb{Z}} \mathbb{Q}_p$. This will generalize well to $\mathbb{T}$, because to give a maximal ideal of $\mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is still the same as giving a continuous homomorphism $\mathbb{T} \to \overline{\mathbb{Z}}_p$.

This discussion of the ideals can be summarized in the following proposition.

Proposition 3.1. We have correspondences

$$\{\text{maximal ideals of } \mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}_p\} \longleftrightarrow \left\{\text{systems of Hecke eigenvalues } \lambda : \mathbb{T} \to \overline{\mathbb{Z}}_p\right\},$$

and

$$\{\text{maximal primes of } \mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Z}_p\} \longleftrightarrow \{\text{maximal ideals of } \mathbb{T} \otimes_{\mathbb{Z}} \mathbb{F}_p\} \longleftrightarrow \left\{\text{systems of Hecke eigenvalues } \lambda : \mathbb{T} \to \overline{\mathbb{F}}_p\right\}.$$

Regarding the structure of $\mathbb{T}$, we have the following theorem which we will present without proof (see Theorem 2.7 of [E]). It resembles theorem 2.1 from the previous section, but its proof involves going through the theory of Galois deformations and universal deformation rings.

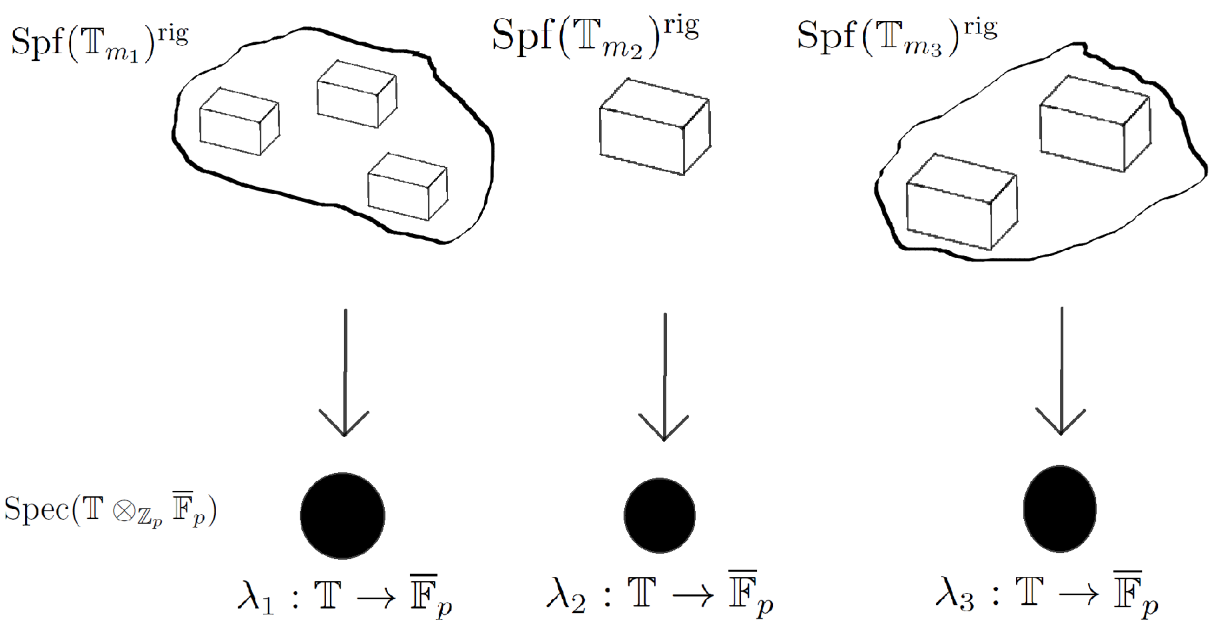
Theorem 3.1. *$\mathbb{T}$ is Noetherian and has finitely many maximal ideals. We have $\mathbb{T} \cong \prod_m \mathbb{T}_m$. It has Krull dimension ≥ 3 over $\mathbb{Z}_p$.*

At least in some cases, it should be known how the $\mathbb{T}_m$ actually look like.

Example 3.1. In the case where $N = 49$ and $p = 3$, there is a universal deformation space corresponding to residual representation coming from $\overline{\rho} : G_{\mathbb{Q},\{3,7,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_3)$, coming from the 3-torsion points of the elliptic curve $X_0(7^2)$. Since $R = T$ theorems seem to be known in this case, one knows this elliptic curve comes from a modular form of level $N = 49$ and weight 2, and that for the maximal ideal m corresponding to the induced $\overline{\mathbb{F}}_3$-valued system of Hecke eigenvalues of $\mathbb{T}$, one has $\mathbb{T}_m \cong R_{\overline{\rho}}^{\mathrm{univ}}$. This is computed by Mazur in the yellow book to be isomorphic to $\mathbb{Z}_3[[t_1, t_2, t_3, t_4]]/((1 + t_4)^3 - 1)$. Upon inverting 3, this becomes simpler, and we see that $\mathbb{T}_m[1/3] \cong \mathbb{Z}_3[[t_1, t_2, t_3]] \oplus \mathbb{Z}_3[\mu_3][[t_1, t_2, t_3]]$. At least in terms of $\overline{\mathbb{Z}}_3$-valued points, we see that $(\mathrm{Spf}\mathbb{T}_m)^{\mathrm{rig}}$ is the union of three open polydiscs. These can also be visualized as being cubes.

In general, we have the following intuition. We think of the formal scheme $\mathrm{Spf}\mathbb{T}$ as being the union of the finitely many formal schemes $\mathrm{Spf}\mathbb{T}_m$. The associated rigid space $(\mathrm{Spf}\mathbb{T})^{\mathrm{rig}}$ has closed points corresponding to $\overline{\mathbb{Z}}_p$-valued Hecke eigensystems of $\mathbb{T}$; each of these reduce to one of finitely many $\overline{\mathbb{F}}_p$-valued Hecke eigensystems of $\mathbb{T}$. Each $(\mathrm{Spf}\mathbb{T}_m)^{\mathrm{rig}}$ is conjecturally 3-dimensional and perhaps a locally complete intersection.

The picture I would like to keep in mind for this situation in general should be something of the following. One thinks of $(\mathrm{Spf}\mathbb{T})^{\mathrm{rig}}$ as being a rigid space whose points correspond $\overline{\mathbb{Z}}_p$-valued systems of Hecke eigenvalues. It is naturally grouped into the finitely many subspaces different $(\mathrm{Spf}\mathbb{T}_m)^{\mathrm{rig}}$. These different subspace are parametrized by the different $\overline{\mathbb{F}}_p$-valued systems of Hecke eigenvalues, and each such subspaces encompass these $\overline{\mathbb{Z}}_p$-valued systems which reduce to their corresponding $\overline{\mathbb{F}}_p$-valued systems. The situation is depicted in the picture below.



The relation to the previous picture is as follows. As k becomes larger in $\mathbb{T}_{\leq k}^{(p)} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, we are adding more and more local components in the previous picture, while the number of black dots is constant. In the limit, the local components fill up the space in the cubes to make $(\mathrm{Spf}\mathbb{T}_m)^{\mathrm{rig}}$ a 3-dimensional space.

References

[AM] Atiyah-Macdonald, Introduction to commutative algebra.
[E] Emerton, p-adic families of modular forms