THE RIEMANN-ROCH THEOREM

GAL PORAT

Abstract. These are notes for a talk which introduces the Riemann-Roch Theorem. We present the theorem in the language of line bundles and discuss its basic consequences, as well as an application to embeddings of curves in projective space.

1. Algebraic Curves and Riemann Surfaces

A deep theorem due to Riemann says that every compact Riemann surface has a nonconstant meromorphic function. This leads to an equivalence

$$\{\text{smooth projective complex algebraic curves}\} \longleftrightarrow \{\text{compact Riemann surfaces}\}.$$

Topologically, all such Riemann surfaces are orientable compact manifolds, which are all genus g surfaces.

Example 1.1. (1) The curve $\mathbb{P}^1_{\mathbb{C}}$ corresponds to the Riemann sphere.
(2) The (smooth projective closure of the) curve $\mathcal{C} : y^2 = x^3 - x$ corresponds to the torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

Throughout the talk we will frequentely use both the point of view of algebraic curves and of Riemann surfaces in the context of the Riemann-Roch theorem.

2. Divisors

Let $\mathcal{C}$ be a complex curve. A divisor of $\mathcal{C}$ is an element of the group $\mathrm{Div}(\mathcal{C}) := \oplus_{P\in\mathcal{C}}\mathbb{Z}$. There is a natural map $\deg : \mathrm{Div}(\mathcal{C}) \to \mathbb{Z}$ obtained by summing the coordinates.

Functions $f \in K(C)^\times$ give rise to divisors on $\mathcal{C}$. Indeed, there is a map $\mathrm{div} : K(\mathcal{C})^\times \to \mathrm{Div}(\mathcal{C})$, given by setting

$$\mathrm{div}(f) = \sum_{P\in\mathcal{C}} v_P(f)P.$$

One can prove that $\deg(\mathrm{div}(f)) = 0$ for $f \in K(\mathcal{C})^\times$; essentially this is a consequence of the argument principle, because the sum over all residues of any differential of a compact Riemann surface is equal to 0.

Similarly, given a meromorphic differential ω of $\mathcal{C}$, one can define a divisor $\mathrm{div}(\omega) \in \mathrm{Div}(\mathcal{C})$, using the same formula. However, it is no longer true that $\deg(\mathrm{div}(\omega)) = 0$; for example, the differential $\omega = dz$ on $\mathbb{P}^1_{\mathbb{C}}$ has $\mathrm{div}(\omega) = -2\infty$.

To a divisor D we can associate a line bundle $\mathcal{L}(D)$ on $\mathcal{C}$, defined by

$$\mathcal{L}(D)(U) = \big\{ f \in K(C)^\times : \mathrm{div}(f)_P \geq -D_P \text{ for } P \in U \big\} \cup \{0\},$$

and a sheaf of differentials on $\mathcal{C}$, given by

$$\Omega^1(D)(U) = \{\omega \text{ meromorphic differential} : \mathrm{div}(\omega)_P \geq -D_P \text{ for } P \in U\} \cup \{0\}.$$

In particular, we note that $\mathcal{L}(0) = \mathcal{O}_{\mathcal{C}}$ (the sheaf of functions of $\mathcal{C}$) and $\Omega^1(0) = \Omega^1_{\mathcal{C}}$ (the sheaf of 1-forms of $\mathcal{C}$).

Moreover, note that given ω a meromorphic differential, we have an isomorphism $\mathcal{L}(D + \mathrm{div}(\omega)) \xrightarrow{\sim} \Omega^1(D)$, given by $f \mapsto f\omega$.

3. The Riemann-Roch Theorem

We are finally ready to state the Riemann-Roch theorem.

Theorem 3.1. *Let $\mathcal{C}$ be a smooth projective curve, and let $D \in \mathrm{Div}(\mathcal{C})$. Let g be the genus of $\mathcal{C}$, thought of as a Riemann surface. Then*

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) - \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \Omega^1(-D)) = \mathrm{deg} D + 1 - g.$$

Before presenting a sketch of a proof, let us make some remarks and present some examples.

Remark 3.1. (1) The dimensions $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D))$ and $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \Omega^1(-D))$ are finite because $\mathcal{L}(D)$ and $\Omega^1(D)$ are coherent sheaves on $\mathcal{C}$.

(2) Since $\mathcal{C}$ is compact, $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(0)) = \mathrm{H}^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = \mathbb{C}$. Plugging $D = 0$ in the Riemann-Roch theorem then gives $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \Omega^1_{\mathcal{C}}) = \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \Omega^1(0)) = g$. In some sense, this is a shade of the Hodge decomposition theorem for curves, which states that Poincare duality induces an isomorphism $\mathrm{H}^0(\mathcal{C}, \Omega^1_{\mathcal{C}}) \oplus \overline{\mathrm{H}^0(\mathcal{C}, \Omega^1_{\mathcal{C}})} \cong \mathrm{H}^1(\mathcal{C}, \mathbb{C})$. The latter singular cohomology $\mathrm{H}^1(\mathcal{C}, \mathbb{C})$ is more familiar and indeed has dimension $2g$, as $\mathcal{C}$ is topologically an orientable surface of genus g.

(3) Taking $D = \mathrm{div}(\omega)$ for any meromorphic differential in the equation gives

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(\mathrm{div}(\omega))) - \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \Omega^1(-\mathrm{div}(\omega))) = \mathrm{deg}\mathrm{div}(\omega) + 1 - g,$$

but $\Omega^1(-\mathrm{div}(\omega)) \cong \mathcal{L}(0) = \mathcal{O}_{\mathcal{C}}$, and $\mathcal{L}(\mathrm{div}(\omega)) \cong \Omega^1_{\mathcal{C}}$, so according to (2), the left hand side is $g - 1$. Therefore

$$\mathrm{deg}\mathrm{div}(\omega) = 2g - 2,$$

which is the Euler characteristic of $\mathcal{C}$.

(4) If $\mathrm{deg} D > 2g-2$, then $\mathrm{deg}(\mathrm{div}(\omega) - D) < 0$. This implies that $\Omega^1(-D) \cong \mathcal{L}(\mathrm{div}(\omega) - D) = 0$, and the equation becomes

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) = \mathrm{deg} D + 1 - g.$$

This equation shows how to a large extent the geometry of a curve is controled by its genus, which is only a topological invariant. More precisely, there tend to be less meromorphic functions with a prescribed set of poles as g becomes larger, and the exact amount to which the dimension of these spaces decreases is g (at least, if $\mathrm{deg} D$ is large compared to g).

(5) In fact the equation in (4) is even more impressive when one considers how nontrivial it is to show in general that a Riemann surface has any meromorphic function.

Next let us consider some explicit examples.

Example 3.1. (1) For $\mathcal{C} = \mathbb{P}^1_{\mathbb{C}}$, which is the Riemann sphere, we have $g = 0$ and $2g-2 = -2$. So (4) says that if $\deg D > -2$ then $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) = \deg D + 1$.

For example, let's consider $D = n\mathcal{O} \in \mathrm{Div}(\mathbb{P}^1_{\mathbb{C}})$, where $\mathcal{O}$ is the origin. Then $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D))$ is the space of meromorphic functions on $\mathbb{P}^1_{\mathbb{C}}$ which are allowed to have a pole of order at most n at $\mathcal{O}$, and which have no other poles. Then one sees that

$$\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) = \mathrm{span}_{\mathbb{C}} \left\{ 1, \frac{1}{z}, \frac{1}{z^2}, ..., \frac{1}{z^n} \right\},$$

and this indeed has dimension $n + 1 = \deg D + 1$.

(2) For the projective curve defined by $\mathcal{C} : y^2 = x^3 - x$, one has $g = 1$, because the associated Riemann surface $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ is a torus. So $2g-2 = 0$, and (4) is saying that if $\deg D > 0$ then $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) = \deg D$. For example, let $\mathcal{O}$ be the unique point of $\mathcal{C}$ at infinity (given by $[0 : 1 : 0]$ in projective coordinates), and take $D = n\mathcal{O}$. Then $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) = n$, and indeed we find out that

$$\begin{aligned}
\mathrm{H}^0(\mathcal{C}, \mathcal{L}(\mathcal{O})) &= \mathrm{span}_{\mathbb{C}}\{1\}, \\
\mathrm{H}^0(\mathcal{C}, \mathcal{L}(2\mathcal{O})) &= \mathrm{span}_{\mathbb{C}}\{1, x\}, \\
\mathrm{H}^0(\mathcal{C}, \mathcal{L}(3\mathcal{O})) &= \mathrm{span}_{\mathbb{C}}\{1, x, y\}, \\
\mathrm{H}^0(\mathcal{C}, \mathcal{L}(4\mathcal{O})) &= \mathrm{span}_{\mathbb{C}}\{1, x, y, x^2\}, \\
\mathrm{H}^0(\mathcal{C}, \mathcal{L}(5\mathcal{O})) &= \mathrm{span}_{\mathbb{C}}\{1, x, y, x^2, xy\}, \\
\mathrm{H}^0(\mathcal{C}, \mathcal{L}(6\mathcal{O})) &= \mathrm{span}_{\mathbb{C}}\{1, x, y, x^2, xy, x^3\}.
\end{aligned}$$

(y^2 may seem to be missing from the last space, but it is not since $y^2 = x^3 - x$!). In fact, reversing this kind of calculation can be used to prove that any smooth projective complex curve of genus 1 has a Weierstrass form, i.e. can be put in the form $y^2 = f(x)$ where $f(x)$ is a cubic. In particular, this shows that any genus 1 curve can embedded into $\mathbb{P}^2_{\mathbb{C}}$.

Finally, let's sketch a proof of the theorem. One needs two major pieces of input which we shall regard as black boxes:

(1) The aforementioned fact that $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \Omega^1_{\mathcal{C}}) = g$, and

(2) Serre duality, which says in our context that if $\mathcal{F}$ is a vector bundle on $\mathcal{C}$, then $\mathrm{H}^0(\mathcal{C}, \Omega^1_{\mathcal{C}} \otimes \mathcal{F}^\vee) \cong \mathrm{H}^1(\mathcal{C}, \mathcal{F})$ (here H^1 means coherent cohomology; but only the formal properties of this cohomology are important for our proof).

Then (2) implies that $\mathrm{H}^0(\mathcal{C}, \Omega^1_{\mathcal{C}}(-D)) \cong \mathrm{H}^1(\mathcal{C}, \mathcal{L}(D))$, so the equality we want to prove is

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) - \dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{C}, \mathcal{L}(D)) = \deg D + 1 - g.$$

This is easier to handle, because if we let $\chi(D) := \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) - \dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{C}, \mathcal{L}(D))$, this quantity behaves like an Euler characteristic, and in particular is additive with respect to D. Using this formal property it is not hard to verify that $\chi(D) = \chi(0) + \deg D$, which allows us to reduce to the case where $D = 0$; in other words, we need to prove that

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) - \dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = 1 - g.$$

As $\mathcal{C}$ is projective, we have $\mathrm{H}^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = \mathbb{C}$, and it remains to show $\dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = g$. This follows from (1) and (2), because $\mathrm{H}^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \cong \mathrm{H}^0(\mathcal{C}, \Omega^1_{\mathcal{C}})$.

4. Application: projective embeddings of curves

Let X be a complex variety and $\mathcal{L}$ a line bundle on X. Any $n+1$ global sections $s_0, ..., s_n \in \mathrm{H}^0(X, \mathcal{L})$ give rise to a rational map

$$X \to \mathbb{P}^n_{\mathbb{C}}, \; x \mapsto [s_0(x) : ... : s_n(x)]$$

(at least if $s_0, ..., s_n$ are not all identically 0). If there is no x such that $s_0(x) = ... = s_n(x) = 0$ (possibly after scaling), we see that this rational map is actually a morphism. If such $s_0, ..., s_n$ exist, we say that $\mathcal{L}$ is *generated by global sections*.

If there is a choice of $s_0, ..., s_n$ such that the induced morphism is moreover a closed immersion, we say that $\mathcal{L}$ is *very ample*. If there is some $n > 0$ so that $\mathcal{L}^{\otimes n}$ is very ample, we say that $\mathcal{L}$ is ample.

The point is that $\mathcal{L}$ has to have many functions in an appropriate sense in order to define an embedding of X into projective space.

Suppose now that $X = \mathcal{C}$ is a smooth projective complex curve. By definition it can be embedded into projective space, but can we be more precise? Given $D \in \mathrm{Div}(\mathcal{C})$, we can ask whether $\mathcal{L}(D)$ is very ample, i.e. if it gives rise to an embedding into projective space.

Example 4.1. Let $\mathcal{C} : y^2 = x^3 - x$ as before, and suppose $D = \mathcal{O}$, where $\mathcal{O} \in \mathcal{C}$ is the point at infinity. Then $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D))$ consists only of the constant functions, as otherwise we would have a degree 1 function $\mathcal{C} \to \mathbb{P}^1_{\mathbb{C}}$, which is impossible as $\mathcal{C}$ has genus 1; therefore any morphism $X \to \mathbb{P}^n_{\mathbb{C}}$ rising from $\mathcal{L}$ is constant, and in particular cannot be an embedding.

However, D is ample, because $3D = 3\mathcal{O}$ is very ample. Indeed, $x, y, 1 \in \mathrm{H}^0(X, \mathcal{L}(3\mathcal{O}))$ and $\mathcal{C} \to \mathbb{P}^2$ gives rise to the defining embedding of $\mathcal{C}$ in $\mathbb{P}^2$.

Given a curve, we may ask: which line bundles are generated by global sections? Which are very ample? Which are ample?

Theorem 4.1. *Let* $D \in \mathrm{Div}(\mathcal{C})$.

(1) $\mathcal{L}(D)$ *is generated by global sections if and only if for every* $P \in \mathcal{C}$*, we have* $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D - P)) = \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) - 1$.

(2) $\mathcal{L}(D)$ *is very ample if and only if for every two points* $P, Q \in \mathcal{C}$*, we have* $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D - P - Q)) = \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) - 2$.

Proof. We prove only (1); (2) is similar once some setup is in place, but it is a bit of a detour for us. Let n_P be the coefficient of P in D; we have a natural inclusion $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D - P)) \hookrightarrow \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D))$, and the set of functions in $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D))$ with a pole of order n_P at P span the cokernel. If there is no such function then this precisely means P is a base point of D, and in that case $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D - P)) = \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D))$. Otherwise, the inclusion $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D - P)) \hookrightarrow \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D))$ is strict, so there is a function with a pole of order n_P at P, but every two such functions are linearly dependant mod $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D - P))$, so $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D - P)) = \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) - 1$. □

Corollary 4.1. *(1) If* $\deg D \geq 2g$*, then* $\mathcal{L}(D)$ *is base point free.*

(2) If $\deg D \geq 2g + 1$*, then* $\mathcal{L}(D)$ *is very ample.*

(3) $\deg D > 0$ *if and only if* D *is ample.*

Proof. For (1), we have $\deg(D-P), \deg D > 2g-2$, so by Riemann-Roch, $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D-P)) = \deg(D-P)+1-g$ and $\mathrm{H}^0(\mathcal{C}, \mathcal{L}(D)) = \deg(D)+1-g$ and the result follows from part (1) of the previous theorem. For (2) we argue similarly.

For (3), if $\deg D > 0$ then for some n we have $\deg(nD) \geq 2g+1$, so by (2) $\mathcal{L}(nD) = \mathcal{L}(D)^{\otimes n}$ is very ample, proving that D is ample. Conversely, if D is ample, that means $\mathcal{L}(nD)$ is very ample for some n, so it is enough to show that $\deg D > 0$ if D is very ample. Well, if D is very ample then $\mathcal{L}(D)$ is the pullback of $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$ via an induced map $f : \mathcal{C} \to \mathbb{P}^n_{\mathbb{C}}$, so $\deg D = \deg f^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1) = \deg f > 0$. □