RELATING FIELDS OF NORMS ASSOCIATED WITH A FINITE UNRAMIFIED EXTENSION

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Let L/K be a finite unramified extension of nonarchimedean local fields of characteristic zero, and let l, k be the corresponding residue fields. We show how to choose primes $\pi_K \in K, \pi_L \in L$ for which we are able to make explicit the relation between the associated fields of norms $E_{L,\pi_L} \simeq l((X))$ and $E_{K,\pi_K} \simeq k((X))$.

1 Introduction

Let K be a finite extension of $\mathbb{Q}_p$, and let $\pi \in K$ be a prime element. We let $\mathcal{F}_\pi$ denote the set of Frobenius power series which start with π, i.e.

$$\mathcal{F}_\pi = \{\phi \in \mathcal{O}_K[[X]] | \phi = \pi X + ..., \phi \equiv X^q (\text{mod } \pi\mathcal{O}_K[[X]])\}$$

By Lubin-Tate theory, any $\phi \in \mathcal{F}_\pi$ determines a unique commutative formal group law F_ϕ for which ϕ is a homomorphism. With respect to the addition defined by F_ϕ, one has corresponding $\mathcal{O}_K$-modules

$$W_\phi^n = \{x \in \mathbb{C}_p, |x| < 1 | \phi^n(x) = 0\}$$

Then $K_\pi = K(\cup_{n\geq 0} W_\phi^n)$ is a maximal totally ramified extension in K^{ab} which depends only on π and not on ϕ.

There is also the Tate module

$$T_\phi = \lim_{\leftarrow}(... \xrightarrow{\phi} W_\phi^2 \xrightarrow{\phi} W_\phi^1 \xrightarrow{\phi} 0)$$

Using the above Tate module, for each prime $\pi \in K$ one can construct a field of norms $E_{K,\pi}$ as presented in [3]. First, one constructs the tilting $\mathbb{C}_p^\flat$, i.e. the fraction field of

$$\mathcal{O}_{\mathbb{C}_p^\flat} := \lim_{\substack{\leftarrow \\ x^q \leftarrow x}} \mathcal{O}_{\mathbb{C}_p^\flat}/\pi\mathcal{O}_{\mathbb{C}_p^\flat} = \lim_{\leftarrow}(... \xrightarrow{x\mapsto x^q} \mathcal{O}_{\mathbb{C}_p^\flat}/\pi\mathcal{O}_{\mathbb{C}_p^\flat} \xrightarrow{x\mapsto x^q} \mathcal{O}_{\mathbb{C}_p^\flat}/\pi\mathcal{O}_{\mathbb{C}_p^\flat})$$

It is clearly a k-algebra, and when induced with the right norm it is a complete nonarchimedean field of characteristic p; see Proposition 1.4.7 in [3]. Now one picks any generator $(..., \omega_2, \omega_1, 0)$ of T_ϕ, and after one checks that $(..., \overline{\omega_2}, \overline{\omega_1}, 0) \in \mathcal{O}_{\mathbb{C}_p^\flat}$, the embedding

$$k((X)) \mapsto \mathbb{C}_p^\flat, f(X) \mapsto f(\omega)$$

Becomes well defined. It can be checked that the image of this map is a subfield $E_{K,\pi} \subset \mathbb{C}_p^\flat$ that is preserved by the action of $\mathrm{Gal}(K_\pi/K)$ and depends only on π, and not on the choice of the generator or ϕ. However, the isomorphism

$$\eta_{\phi,\omega} : k((X)) \xrightarrow{\sim} E_{K,\pi}$$

Still depends on the choice of a Frobenius power series ϕ and a generator $\omega \in T_\phi$.

In fact, $E_{K,\pi}$ is a subfield of the tilting $\widehat{K_\pi}^\flat \subset \mathbb{C}_p^\flat$. It can be shown that under the topology induced from $\mathbb{C}_p^\flat$, $E_{K,\pi} \simeq k((X))$ as topological fields where the latter is induced with X-topology, and then we have $\widehat{E_{K,\pi}^{perf}} = \widehat{K_\pi}^\flat$.

Now suppose that L/K is a finite unramified extension. If $\pi_K \in K$ and $\pi_L \in L$ are primes such that $K_{\pi_K} \subset L_{\pi_L}$, it follows that

$$E_{K,\pi_K} \subset \widehat{E_{L,\pi_L}^{perf}}$$

Since $E_{L,\pi_L} \simeq l((X))$, we have

$$\widehat{E_{L,\pi_L}^{perf}} \simeq \cup_{n\geq 0} \widehat{l((X^{q^{-n}}))}$$

Therefore, for any choice of primes π_K, π_L with $K_{\pi_K} \subset L_{\pi_L}$, Frobenius power series $\phi_K \in \mathcal{F}_{\pi_K}, \phi_L \in \mathcal{F}_{\pi_L}$ and Tate modules generators $\omega_K \in T_{\phi_K}, \omega_L \in T_{\phi_L}$, one obtains a commutative diagram

$$\begin{array}{ccc} k((X)) & \xrightarrow{X \mapsto g(X)} & \cup_{n\geq 0} \widehat{l((X^{q^{-n}}))} \\ \downarrow{\scriptstyle \eta_{\phi_K,\omega_K}} & & \downarrow{\scriptstyle \eta_{\phi_L,\omega_L}} \\ E_{K,\pi_K} & \xrightarrow{\subset} & \widehat{E_{L,\pi_L}^{perf}} \end{array} \quad (1)$$

The aim of this note is to find some specific $\pi_K, \phi_K, \omega_K, \pi_L, \phi_L, \omega_L$ for which an explicit element $g(X) \in \cup_{n\geq 0} \widehat{l((X^{q^{-n}}))}$ can be computed, thus making the containment of E_{K,π_K} in $\widehat{E_{L,\pi_L}^{perf}}$ explicit.

2 The π's, ϕ's and ω's

In this section we aim the to find our desired $\pi_K, \phi_K, \omega_K, \pi_L, \phi_L, \omega_L$ for which an explicit $g(X) \in \cup_{n\geq 0} \widehat{l((X^{q^{-n}}))}$ can be computed. We will show how to construct these elements and prove the relevant properties that will be needed for the next sections.

2.1 Choosing the π's, ϕ's and ω's

The following definition and two lemmas are presented in sections 0.2 and 0.3 of [1], although the notations differ.

Definition 1. A **Lubin-Tate splitting** (over K) is a choice of a Frobenius automorphism $\varphi_K \in \mathrm{Gal}(\overline{K}/K)$, that is, an extension of the Frobenius $\varphi_K \in \mathrm{Gal}(K^{nr}/K)$ to an automorphism of $\overline{K}$.

We let $K_{\varphi_K} := \overline{K}^{\varphi_K}$.

Lemma 2. *Fix a Lubin–Tate splitting φ_K. There exists a unique norm-compatible sequence $\{\pi_E \in E^\times | K \subset E \subset K_{\varphi_K}, [E : K] < \infty\}$.*

In fact, the construction of the proof shows more specifically that π_E is the unique universal norm from the maximal totally ramified extension $E^{ab} \cap K_{\varphi_K}$, and so $E_{\pi_E} = E^{ab} \cap K_{\varphi_K}$. Similarly, this also determines all the class fields with norm groups $\langle \pi_E \rangle U_n(E)$. We change notations and denote the former by E_∞ and the latter by E_n.

We now fix φ_K, so that the unique norm-compatible sequence of primes is determined.

Lemma 3. *Fix a Lubin-Tate splitting as above, and let $E \subset K_{\varphi_K}$ with $[E : K] < \infty$. There is a unique Frobenius power series $\phi_E(X) \in \mathcal{F}_{\pi_E}$ such that*

$$\phi_E(\pi_{E_{n+1}}) = \phi(\pi_{E_n}), n \geq 1 \text{ and } \phi_E(\pi_{E_1}) = 0$$

Hence both ϕ_K and $\omega_K = (, ..., \pi_{K_1}, 0)$ are determined.

Now suppose that L/K is an unramified extension with $[L : K] = d$. If φ_K is a Lubin-Tate splitting of K then $\varphi_L := \varphi_K^d$ is a Lubin-Tate splitting of L. Thus, *any choice of a Lubin-Tate splitting $\varphi_K \in G_K$ determines a unique choice of $\pi_K, \phi_K, \omega_K, \pi_L, \phi_L, \omega_L$.*

However, for the sequel the primes π_{E_n} and Frobenius power series ϕ_E above will also be important. Since L/K is an unramified extension and the lemmas above only refers to totally ramified extensions of the base field, there will be no ambiguity when we write π_{E_n} or π_E for a field E that contains L.

2.2 Relevant Properties

Let L/K be a finite unramified extension, and fix a choice of a Lubin-Tate splitting φ_K of K. By subsection 2.1 this determines a Lubin-Tate splitting φ_L of L and unique norm-compatible choices of primes $\pi_E \in E$ for $K \subset E \subset K_{\varphi_K}$ or $L \subset E \subset L_{\varphi_L}$.

For an integer d, we let $E_{n,d} := E_n E_d^{nr}$, with E_d^{nr} the unique unramified extension of E of degree d.

Our main aim in this section is proving Proposition 6. For this we will need the following lemma on Lubin-Tate splittings.

Lemma 4. *(i) Let F/K be a finite Galois extension with $L \subset F \subset L_{\varphi_L}$. Then $\pi_F \in K_{\varphi_K}$, i.e. π_F is fixed by φ_K.*

(ii) Let E/K a finite extension with $K \subset E \subset K_{\varphi_K}$. Then $\pi_{E_d^{nr}} = \pi_E$.

Proof of (i). Since F/K is Galois, φ_K induces an automorphism of of F. Denoting by $\widehat{F^\times}$ the profinite completion of $F^\times$, we have by Theorem 6.11 in [2] the commutative diagram

$$\begin{array}{ccc} \widehat{F^\times} & \xrightarrow{\sim} & \mathrm{Gal}(F^{ab}/F) \\ \downarrow{\scriptstyle \varphi_K} & & \downarrow{\scriptstyle \gamma \mapsto \varphi_K \circ \gamma \circ \varphi_K^{-1}} \\ \widehat{F^\times} & \xrightarrow{\sim} & \mathrm{Gal}(F^{ab}/F) \end{array} \tag{2}$$

With the horizontal maps being the Artin reciprocity map.

As we have seen in subsection 2.1, π_F is the unique prime corresponding to the maximal totally ramified abelian extension $F_\infty = F^{ab} \cap L_{\varphi_L}$ of F. This means that π_F is sent to $\varphi_L|_{F^{ab}}$ by the Artin reciprocity map.

Therefore, on the right-down path, we have $\pi_F \mapsto \varphi_L \mapsto \varphi_L$. On the other hand, π_F is sent to $\varphi_K(\pi_F)$ by the left vertical map. By injectivity of the reciprocity map we must have $\pi_F = \varphi_K(\pi_F)$.

Proof of (ii). Since $E \subset K_{\varphi_K}$, for any $n \geq 0$ we have the corresponding totally ramified abelian extensions $E \subset E_n \subset E_\infty := E^{ab} \cap K_{\varphi_K}$. We now consider the field $E_{n,d}$, which is clearly contained in L_{φ_L}. Because $K_{\varphi_K} = E_{\varphi_K}$ and $E_{n,d}/E$ is Galois, we may use part (i) (substituting $E_{n,d}$ for F, E for K and E_f for L) to conclude that $\pi_{E_{n,d}}$ is fixed by φ_K. In particular, $\pi_{E_{n,d}} \in E_n$.

We now have the cartesian diagram

$$\begin{array}{ccccc} & & E_{n,d} & & \\ & / & & \backslash & \\ E_n & & & & E_n \\ & \backslash & & / & \\ & & E & & \end{array} \tag{3}$$

And restriction induces an isomorphism

$$\mathrm{Gal}(E_{n,d}/E_d^{nr}) \xrightarrow{\sim} \mathrm{Gal}(E_n/E)$$

Using the norm compatibility we have

$$\pi_{E_n} = N_{E_{n,d}/E_d^{nr}}(\pi_{E_{n,d}}) = \prod_{\sigma \in \mathrm{Gal}(E_{n,d}/E_d^{nr})} \pi_{E_{n,d}} = \prod_{\sigma \in \mathrm{Gal}(E_n/E)} \pi_{E_{n,d}} = N_{E_n/E}(\pi_{E_{n,d}})$$

Where for the third equality we recall that $\pi_{E_{n,d}} \in E_n$.

In particular, for $n = 0$ we have $\pi_{E_d^{nr}} \in E$. The above equality shows then that $\pi_{E_d^{nr}}$ is a universal norm from $E_\infty = E^{ab} \cap K_\phi$, and hence $\pi_{E_d^{nr}} = \pi_E$ by the discussion after Lemma 2. $\square$

Proposition 5. *(i)* $K_n \subset L_n$ *for all* $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

(ii) Let $d = [L : K]$. *Then* $K_{n,d} \subset L_n$ *and* $N_{L_n/K_{n,d}}(\pi_{L_n}) = \pi_{K_n}$*; in particular,* $\pi_L = \pi_K$.

(iii) L_n/K *is Galois.*

(iv) ϕ_L *is fixed by* φ_K. *In other words,* $\phi_L \in \mathcal{O}_K[[X]]$.

Proof of (i). First, I claim that $K_\infty \subset L_\infty$. Indeed, we have $K_\infty = K^{ab} \cap K_{\varphi_K} \subset L^{ab} \cap K_{\varphi_K} \subset L^{ab} \cap L_{\varphi_L} = L_\infty$. From local class field theory we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Gal}(L_\infty/L) & \xleftarrow{\sim} & \mathrm{Gal}(L^{ab}/L^{nr}) & \xrightarrow{\sim} & U(L) \\ \downarrow{\scriptstyle Res} & & \downarrow{\scriptstyle Res} & & \downarrow{\scriptstyle N_{L/K}} \\ \mathrm{Gal}(K_\infty/K) & \xleftarrow{\sim} & \mathrm{Gal}(K^{ab}/K^{nr}) & \xrightarrow{\sim} & U(K) \end{array} \tag{4}$$

The left square is clearly commutative since all the maps are restrictions. The right square is commutative by local class field theory.

For any n, the subgroup $\mathrm{Gal}(L_\infty/L_n) \triangleleft \mathrm{Gal}(L_\infty/L)$ corresponds to the subgroup $U_n(L) \triangleleft U(L)$, and similarly for K. The map N sends $U_n(L)$ into $U_n(K)$ (see Proposition 1 in Chapter V, §2 of [4]), so the map $\mathrm{Gal}(L_\infty/L) \to \mathrm{Gal}(K_\infty/K)$ sends $\mathrm{Gal}(L_\infty/L_n)$ to $\mathrm{Gal}(K_\infty/K_n)$. This means that any automorphism of L_∞ the fixes L_n must fix K_n as well. In other words, $K_n \subset L_\infty^{\mathrm{Gal}(L_\infty/L_n)} = L_n$.

Proof of (ii). Since the extensions L_n/L and K_n/K are totally ramified and the extension L/K is unramified of degree d, we see that L_n/K_n has inertia degree d, and therefore $K_{n,d} \subset L_n$.

In particular, we have $L \subset K_{n,d} \subset L_{\varphi_L}$ and so $\pi_{K_{n,d}}$ is defined, and by the norm-compatibility $N_{L_n/K_{n,d}}(\pi_{L_n}) = \pi_{K_{n,d}}$. It remains to show that $\pi_{K_{n,d}} = \pi_{K_n}$. But this follows from Lemma 4, (ii) by taking $E = K_n$.

Proof of (iii). Let $\sigma \in \mathrm{Gal}(\overline{K}/K)$. Since L^{ab}/K is Galois, σ preserves L^{ab} and $\sigma(L_n)$ has norm group $\langle \sigma(\pi_L) \rangle\, U_n(L) = \langle \pi_L \rangle\, U_n(L)$ by part (ii). Therefore $\sigma(L_n) = L_n$, which shows L_n/K is Galois.

Proof of (iv). By Lemma 3, we have that $\phi_L \in \mathcal{O}_L[[X]]$ is is the unique Frobenius power series for π_L with

$$\phi_L(\pi_{L_{n+1}}) = \phi(\pi_{L_n}), n \geq 1 \text{ and } \phi_L(\pi_{L_1}) = 0$$

By part (i) of Lemma 4 and part (iii) we just proved, we know that the π_L, π_{L_n} are all fixed by φ_K. Thus $\phi_L^{\varphi_K}$ satisfies the characterizing properties, since

$$\pi_{L_{n-1}} = \varphi_K(\pi_{L_{n-1}}) = \varphi_K(\phi_L(\pi_{L_n})) = \phi_L^{\varphi_K}(\varphi_K(\pi_{L_n})) = \phi_L^{\varphi_K}(\pi_{L_n})$$

Thus $\phi_L^{\varphi_K} = \phi_L$. $\square$

The final proposition that we will need for the next section is the following.

Proposition 6. *(i) The extension $K_{n,d}/L$ is abelian of degree $(q-1)q^{n-1}$ and*

$$N(K_{n,d}/L) = \langle \pi_L \rangle \, N_{L/K}^{-1}(U_n(K))$$

(ii) $\mathrm{Gal}(L_n/K_{n,d}) \simeq N_{L/K}^{-1}(U_n(K))/U_n(L)$.

Proof of (i). By Proposition 5 (i), we have $K_{n,d} \subset L_n$; because L_n/L is abelian, the subextension $K_{n,d}/L$ is also abelian. We have $[K_{n,d} : L] = [K_n : K] = (q-1)q^{n-1}$.

By local class field theory, we now know that

$$[L^\times : N(K_{n,d}/L)] = [K_{n,d} : L] = (q-1)q^{n-1}$$

Noticing that $N(K_{n,d}/K) = \langle \pi_K^d \rangle \, U_n(K)$ and $N_{K_{n,d}/L}(\pi_{K_{n,d}}) = \pi_L$ by part (ii) of Proposition 5, we have $N(K_{n,d}/L) \subset \langle \pi_L \rangle \, N_{L/K}^{-1}(U_n(K))$.

On the other hand, $N_{L/K}^{-1}(U_n(K))$ is the kernel of the composition $U(L) \xrightarrow{N_{L/K}} U(K) \to U(K)/U_n(K)$. Since L/K is unramified, the map $U(L) \xrightarrow{N_{L/K}} U(K)$ is surjective: see the Corollary to Proposition 3 in Chapter V, §2 of [4]. We therefore obtain

$$U(L)/N_{L/K}^{-1}(U_n(K)) \simeq U(K)/U_n(K)$$

This implies that

$$[L^\times : \langle \pi_L \rangle \, N_{L/K}^{-1}(U_n(K))] = [U(L) : N_{L/K}^{-1}(U_n(K))] = (q-1)q^{n-1}$$

And thus combining this with the equality $[L^\times : N(K_{n,d}/L)] = (q-1)q^{n-1}$ and the inclusion $N(K_{n,d}/L) \subset \langle \pi_L \rangle \, N_{L/K}^{-1}(U_n(K))$, we have

$$N(K_{n,d}/L) = \langle \pi_L \rangle \, N_{L/K}^{-1}(U_n(K))$$

Proof of (ii). This is now immediate because of the correspondence between norm subgroups and abelian extensions. We now know the isomorphisms

$$\mathrm{Gal}(L_n/L) \simeq U(L)/U_n(L), \; \mathrm{Gal}(K_{n,d}/L) \simeq U(L)/N_{L/K}^{-1}(U_n(K)$$

and $\mathrm{Gal}(L_n/K_{n,d})$ is the kernel of the restriction $\mathrm{Gal}(L_n/L) \to \mathrm{Gal}(K_{n,d}/L)$. $\square$

3 Relating E_{K,π_K} and E_{L,π_L}

We continue with the unramified extension L/K of degree d and with the choices of π_K, ϕ_K, ω_K and π_L, ϕ_L, ω_L of section 2. By Proposition 5, part (ii), we have $\pi_K = \pi_L$, so we may denote them both by π. We can then take the tilting construction of $\mathbb{C}_p^\flat$ with respect to π and the q-transition maps or q^d-transition maps. So if we put, as in section 2,

$$\omega_K = (0 + \pi\mathcal{O}_{K_\infty}, \pi_{K_1} + \pi\mathcal{O}_{K_\infty}, \pi_{K_2} + \pi\mathcal{O}_{K_\infty}, ...) \in \widehat{K_\infty}^\flat$$

$$\omega_L = (0 + \pi\mathcal{O}_{L_\infty}, \pi_{L_1} + \pi\mathcal{O}_{L_\infty}, \pi_{L_2} + \pi\mathcal{O}_{L_\infty}, ...) \in \widehat{L_\infty}^\flat$$

In the first case the transition maps are by raising to the q-power, while for the second case the transition maps are given by raising to the q^d-power.

On the other hand, it is known that $\widehat{E_L^{perf}}$ (the completion of the perfection) is equal to $\widehat{L_\infty}^\flat$. Therefore there must be some way to express ω_K as a series in ω_L over l. This is the content of our main theorem.

Theorem 7. *For each n, let A_n be a set of represenatives for $N_{L/K}^{-1}(U_{nd}(K))/U_{nd}(L)$. Define a sequence of series $g_n(X) \in \cup_n l[[X^{-q^n}]]$ by*

$$g_n(X) = \prod_{a \in A_n} \overline{[a]_{\phi_L}}(X^{\frac{1}{q^{dn(d-1)}}})$$

Then the $g_n(X)$ converge to an element $g(X) \in \cup_n \widehat{l[[X^{q^{-n}}]]}$ in the X-topology, and $g(X)$ doesn't depend on any choice of the A_n.

$g(X)$ satisfies

$$g(\omega_L) = \omega_K$$

Proof. Using Proposition 6, (ii) we know that for any n we have $\mathrm{Gal}(L_n/K_{n,d}) \simeq N_{L/K}^{-1}(U_n(K))/U_n(L)$. Thus by Proposition 5,(ii) we have

$$\pi_{K_{nd}} = N_{L_{nd}/K_{nd,d}}(\pi_{L_{nd}}) = \prod_{\sigma \in \mathrm{Gal}(L_{nd}/K_{nd,d})} \sigma(\pi_{L_{nd}}) = \prod_{a \in A_n} [a]_{\phi_L}(\pi_{L_{nd}}) = \left[\prod_{a \in A_n} [a]_{\phi_L}(X)\right]\Bigg|_{X=\pi_{L_{nd}}}$$

Since $[a]_{\phi_L} \in \mathcal{O}_L[[X]]$ for any such a, we may write $\prod_{a \in A_n} [a]_{\phi_L}(X) = \sum_{j=0}^\infty \beta_{j,n} X^j$, with $\beta_{j,n} \in \mathcal{O}_L$. With this notation we have

$$\pi_{K_{nd}} = \sum_{j=0}^\infty \overline{\beta_{j,n}}(\pi_{L_{nd}} \bmod \pi\mathcal{O}_{L_{nd}})^j$$

This sum converges since $\widehat{L_\infty}^\flat$ is complete.

Consider $\omega_K - \sum_{j=0}^{\infty} \overline{\beta_{j,n}} \omega_L^{\overline{\frac{j}{q^{dn(d-1)}}}}$. One can write

$$\omega_K = (0 + \pi\mathcal{O}_{L_\infty}, \pi_{K_d} + \pi\mathcal{O}_{L_\infty}, \pi_{K_{2d}} + \pi\mathcal{O}_{L_\infty}, ...)$$

With the transition maps given by raising to the q^d-power.

Given this convention, the entry at the n'th coordinate is

$$\pi_{K_{nd}} - \sum_{j=0}^{\infty} \overline{\beta_{j,n}} (\pi_{L_{nd}} \bmod \mathcal{O}_{L_{nd}})^j = 0$$

Because taking ω_L to the power of q^{-dk} is the same as shifting by k coordinates to the left. Shifting the nd'th-place $nd - n$ places to the left moves it to the n'th place.

Now $\omega_K - \sum_{j=0}^{\infty} \overline{\beta_{j,n}} \omega_L^{\overline{\frac{j}{q^{dn(d-1)}}}}$ has 0 coefficient in the n'th coordinate, so it must have 0 coefficients in all the lower coordinates as well. This gives

$$|\omega_K - \sum_{j=0}^{\infty} \overline{\beta_{j,n}} \omega_L^{\overline{\frac{j}{q^{dn(d-1)}}}}| \in (\pi^\flat)^{q^{dn}} \mathcal{O}_{\widehat{L_\infty}^\flat} \Rightarrow |\omega_K - \sum_{j=0}^{\infty} \overline{\beta_{j,n}} \omega_L^{\overline{\frac{j}{q^{dn(d-1)}}}}| \leq |\pi^\flat|^{q^{dn}}$$

Notice that $\sum_{j=0}^{\infty} \overline{\beta_{j,n}} \omega_L^{\overline{\frac{j}{q^{dn(d-1)}}}} = g_n(\omega_L)$, so this may be rephrased as

$$|\omega_K - g_n(\omega_L)| \leq |\pi^\flat|^{q^{dn}}$$

We know we can write $\omega_K = \sum \overline{\beta_\alpha} \omega_L^\alpha$ for corresponding $\alpha \in \mathbb{Z}[\frac{1}{q^\infty}]$. Since $|\omega_K - g_n(\omega_L)| \leq |\pi^\flat|^{q^{dn}}$ we must have that up to a certain α, all of the $\overline{\beta_{j,n}}$ are equal to the corresponding $\overline{\beta_\alpha}$, for otherwise $|\omega_K - g_n(\omega_L)|$ would be bounded from below.

This implies that all the coefficients of $g_n(X)$ certain coefficients have to be constant from some point. So $g_n(X) \xrightarrow{n\to\infty} g(X)$ for $g = \sum \overline{\beta_\alpha} X^\alpha$.

Now we have

$$g(\omega_L) = \lim_{n\to\infty} g_n(\omega_L) = \omega_K$$

Where the last equality follows from the bound $|\omega_K - g_n(\omega_L)| \leq |\pi^\flat|^{q^{dn}}$. $\square$

We can say a bit more about this power series.

Proposition 8. *Let $g(X) \in \cup_n \widehat{l[[X^{-q^n}]]}$ be the power series in Theorem 1 such that $g(\omega_L) = \omega_K$. Suppose that q is odd.*

(i) The leading monomial of $g(X)$ is

$$\begin{array}{ll} (-1)^{d-1}X^{\frac{q^d-1}{q^d-q^{d-1}}}, & q \textit{ is odd} \\ X^{\frac{q^d-1}{q^d-q^{d-1}}}, & q \textit{ is even} \end{array}$$

.

(ii) $g(X) \in \cup_n \widehat{\kappa[[X^{-q^n}]]}$.

For the course of the proof, we let $D_n = N_{L/K}^{-1}(U_{nd}(K))/U_{nd}(L)$.

Proof of (i). In order to find the power α of the leading monomial X^α of g, we set A_n as in the notation of the theorem. Note that

$$\#A_n = \#D_n = \#\mathrm{Gal}(L_{nd}/K_{nd,d}) = \frac{q^d-1}{q-1}\frac{q^{d(nd-1)}}{q^{nd-1}} = \frac{q^d-1}{q-1}q^{(nd-1)(d-1)}$$

Thus the smallest exponent for $g(X) = \lim_{n\to\infty} \prod_{a\in A_n} \overline{[a]_{\phi_L}}(X^{\overline{\frac{1}{q^{dn(d-1)}}}})$ is

$$\frac{q^d-1}{q-1}q^{(nd-1)(d-1)}\frac{1}{q^{dn(d-1)}} = \frac{q^d-1}{q^d-q^{d-1}}$$

In order to find the coefficient of $X^{\frac{q^d-1}{q^d-q^{d-1}}}$, we notice that it is the limit $\lim_{n\to\infty}\prod_{a\in A_n} a$.

For a finite abelian group A, one has the easy

$$\prod_{x\in A} x = \prod_{x\in A[2]} x = \left\{ \begin{array}{ll} 0, & \text{if } \#A[2] > 2 \text{ or } \#A[2] = 0 \\ \text{The unique } 0 \neq a \in A[2], & \text{otherwise} \end{array} \right\}$$

We will exploit this identity in order to compute the above limit.

Assume first that q is odd. Let V be the set of q^d-1'th roots of unity in L. For $x \in V$, we have $N_{L/K}(x) = x^{\frac{q^d-1}{q-1}}$, so V^{q-1} is a subset of $N_{L/K}^{-1}(U_{nd}(K))$. In fact, composing with $N_{L/K}^{-1}(U_{nd}(K)) \to D_n$, we remain with a monomorphism

$$V^{q-1} \hookrightarrow D_n$$

Indeed, $V \cap U_{nd}(K) = \{1\}$ since already $V \cap U_1(K) = \{1\}$.

But as $\#D_n = \frac{q^d-1}{q-1}q^{(nd-1)(d-1)}$, we see that

$$D_n[2] = \left\{ \begin{array}{ll} \pm 1, & 2 \text{ divides } \frac{q^d-1}{q-1} \\ 1, & \text{else} \end{array} \right\}$$

Hence

$$\prod_{x \in D_n} x = (-1)^{d-1}$$

So that

$$\prod_{a \in A_n} a \in (-1)^{d-1} U_{nd}(L) \Longrightarrow \prod_{a \in A_n} a \xrightarrow{n \to \infty} (-1)^{d-1}$$

Suppose now that q is even. The homomorphism

$$N_{nd-1} : U_{nd-1}(L)/U_{nd}(L) \to U_{nd-1}(K)/U_{nd}(K)$$

has kernel of cardinality q^{d-1} by (Proposition 2,(ii) in Chapter V, §2 of [4]). As $U_{nd-1}(L)/U_{nd}(L) \simeq \mathbb{F}_{q^d}$ is a 2-torsion abelian group and $ker(N_{nd-1}) \subset D_n$, we see that $\#D_n[2] \geq q^{d-1}$. If $q = 2, d = 1$ the claim is trivial, so we may assume $\#D_n[2] > 2$. Therefore

$$\prod_{x \in D_n} x = 1$$

So in the even case

$$\prod_{a \in A_n} a \in U_{nd}(L) \Longrightarrow \prod_{a \in A_n} a \xrightarrow{n \to \infty} 1$$

Proof of (ii). By part (iv) of Proposition 5, we have $\phi_L = \phi_L^{\varphi_K}$. Recall that $[a]_L$ is characterized by $\phi_L \circ [a]_L = [a]_L \circ \phi_L$, and thus by applying φ_K to both sides we see that

$$\phi_L \circ [a]_L^{\varphi_K} = [a]_L^{\varphi_K} \circ \phi_L$$

Hence $[a]_L^{\varphi_K} = [\varphi_K(a)]_L$. We then have

$$(\prod_{a \in A_n} [a]_L(X))^{\varphi_K} = \prod_{a \in A_n} [\varphi_K(a)]_L(X)$$

Because φ_K preserves both $N_{L/K}^{-1}(U_{nd}(K))$ and $U_{nd}(L)$, it preserves the quotient. So $A_n^{\varphi_K} := \{\varphi_K(a) : a \in A_n\}$ is a set of represenatives for $N_{L/K}^{-1}(U_{nd}(K))/U_{nd}(L)$.

We now see that

$$g(X)^{\varphi_K} = \left[\lim_{n \to \infty} \prod_{a \in A_n} \overline{[a]_L}(X^{\frac{1}{q^{dn(d-1)}}}) \right]^{\varphi_K} = \lim_{n \to \infty} \prod_{a \in A_n^{\varphi_K}} \overline{[a]_{f_L}}(X^{\frac{1}{q^{dn(d-1)}}}) = g(X)$$

Hence $g \in \cup_n \kappa\widehat{[[X^{-q^n}]]}$. $\square$

References

[1] de Shalit E, Koch H.: *Metabelian local class field theory.* Journal für die reine und angewandte Mathematik, Volume: 478, page 85-106 (1996)

[2] Iwasawa K.: *Local class field theory.* Oxford Univ. Press (1986)

[3] Schneider P.: *Galois representations and* (φ, Γ)*-modules.* Cambridge Univ. Press (2017)

[4] Serre J. P.: *Corps Locaux*, Hermann, Paris (1962)