MODULARITY LIFTING THEOREMS AND THE CASE OF GL$_1$

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Abstract. These are notes for a talk which introduces modularity lifting theorems and the Taylor–Wiles-Kisin method. We give a brief introduction to the ideas and work out what the method does in the simplest case, for a Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}_p^\times$ of level 1 and trivial reduction.

1. What is modularity?

Consider the elliptic curve given by

$$E : y^2 + y = x^3 - x^2.$$ 

It is an elliptic curve of conductor 11. If for each $l$ we set $b_l(E) = (l + 1) - \#E(F_l)$, we can check that

$$b_2(E) = -2, b_3(E) = -1, b_5(E) = 1\ldots$$

On the other hand, there is a unique normalized cuspidal newform $f$ of weight 2 and level 11. Its $q$-expansion is given by

$$f(q) = q \prod_{n \geq 1} (1 - q^n)^2 (1 - q^{11n})^2 = \sum_{n \geq 1} a_n(f) q^n.$$ 

Notice that

$$a_2(f) = -2, a_3(f) = -1, a_5(f) = 1\ldots$$

In fact, it turns out that one has an equality

$$a_l(f) = b_l(f)$$

for all primes $l \nmid 11$. This amazing property of the elliptic curve $E$ is what we call “modularity”. More generally, the famous modularity theorem asserts that if $E$ is any elliptic curve over $\mathbb{Q}$ of conductor $N$, there will always exist a cuspidal newform $f$ of weight 2 and level $N$, with the property $a_l(f) = b_l(f)$ for $l \nmid N$. This has a number of consequences having to do with the $L$ function of $E$ which are of great number theoretic interest. For example, the modularity of $E$ implies the Sato-Tate conjecture, which describes how the $b_l$ distribute as $l$ varies.

Modularity can also be considered in a larger scope. For this purpose it will be useful to replace $E$ with its Tate module. Namely, choose a prime $p$ not dividing $N$ and let $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on the Tate module $\lim_{n \to \infty} E[p^n][\overline{\mathbb{Q}}] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. This action is linear, and one obtains a 2-dimensional representation $\rho_E : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Q}_p)$, which satisfies for all $l \nmid Np$ the equality

$$\text{Tr}(\rho(Frob_l)) = b_l(E).$$
Conjecture 1.1. (Fontaine-Mazur). Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Q}_p) \) be a continuous representation satisfying the following properties: \( \rho \) is absolutely irreducible, \( \det(\rho(\text{conj})) = -1 \), \( \text{cond}(\rho) = N \) and \( \rho|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \) is crystalline of Hodge-Tate weights \( \{0, k - 1\} \) where \( k \geq 2 \).

Then there exists a cuspidal newform \( f(q) = \sum_{n \geq 1} a_n(f) q^n \) of weight \( k \) and level \( N \), such that for all primes \( l \nmid Np \), the equality

\[
a_l(f) = \text{Tr}(\rho(\text{Frob}_l))
\]

holds.

 Whenever the condition \( a_l(f) = \text{Tr}(\rho(\text{Frob}_l)) \) holds for almost all \( l \), we will write \( \rho = \rho_f \) (these conditions determine \( \rho \) uniquely). It turns out that for every cuspidal newform \( f \) one can attach a representation \( \rho_f \), and all the conditions appearing in the conjecture are satisfied for \( \rho = \rho_f \). Thus the conjecture is really a surjectivity statement for the assignment \( f \mapsto \rho_f \).

2. \( R = \mathbb{T} \) theorems

We shall now change our point of view once again, in the following way. Instead of trying to show directly that every Galois representation \( \rho \) comes from a newform of a specific weight and level, it will be easier to try and show that a moduli space of Galois representations is equal to a moduli space of such newforms.

“Modularity lifting theorems” are about proving cases of conjecture 1.1 under an additional assumption that \( \overline{\rho} \), the reduction mod \( p \) of \( \rho \), has \( \overline{\rho} = \overline{\rho}_g \) for some modular form \( g \). In light of this, we will now fix a mod \( p \) representation \( \overline{\rho} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_p) \) of conductor \( N \) and \( p \nmid N \).

It turns out there is a moduli space parametrizing the lifts of \( \overline{\rho} \) satisfying the assumptions of 1.1. More precisely, there is a finite \( \mathbb{Z}_p \)-algebra \( R_{N,k}({\overline{\rho}}) \) which satisfies

\[
\{ \text{A - points of } \text{Spec} R_{N,k}({\overline{\rho}}) \} \cong \{ \text{lifts } \rho : \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{A}) \text{ of } \overline{\rho} \text{ of conductor } N, \text{ crystalline at } p \text{ with Hodge-Tate weights } \{0, k - 1\} \}
\]

On the other hand, there’s also a moduli space for the modular forms side. Namely, there exists\(^2\) a finite \( \mathbb{Z}_p \)-algebra \( \mathbb{T}_{N,k}(\overline{\rho}) \) which satisfies

\[
\{ \text{A - points of } \text{Spec} \mathbb{T}_{N,k}(\overline{\rho}) \} \cong \{ \text{newforms } f \text{ of weight } k, \text{ level } N \text{ and } \overline{\rho} = \overline{\rho}_f \} .
\]

The association \( f \mapsto \rho_f \) then yields a closed embedding \( \text{Spec} \mathbb{T}_{N,k}(\overline{\rho}) \hookrightarrow \text{Spec} R_{N,k}({\overline{\rho}}) \), or equivalently a surjection \( R_{N,k}({\overline{\rho}}) \twoheadrightarrow \mathbb{T}_{N,k}(\overline{\rho}) \). The description of the points above then means that modularity is equivalent to this map being an isomorphism! A theorem of this form is called an “\( R = \mathbb{T} \) theorem”.

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\(^1\)Really one should allow to twist \( \rho \) by a finite image character to make this absolutely correct.

\(^2\)For the experts: this is the anemic Hecke algebra localized at the maximal ideal corresponding to \( \overline{\rho} \).
3. The Taylor-Wiles-Kisin method

The $\mathbb{Z}_p$-algebras $T_{N,k}(\overline{\rho})$ and $R_{N,k}(\overline{\rho})$ are finite $\mathbb{Z}_p$-algebras which can have all kind of wild singularities. For example, in some cases\(^3\) they look something like

$$\mathbb{Z}_p[X,Y] / (XY, X + Y - p),$$

which looks like two components glued along the special fiber (after inverting $p$, you have two points; after reducing mod $p$, you only have one point). When trying to prove modularity theorems, the only thing one can control is the dimensions of tangent spaces of these spaces, and they do not tell us so much about these schemes because they can be so singular. The Taylor-Wiles-Kisin method thickens out these algebras to make them smooth, in which case one can make arguments on the level of tangent spaces.

Let’s be more precise. For simplicity, we shall now assume that $N = 1$ (i.e. that $p$ is unramified outside $p$ and that $\rho|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is crystalline), though a more general treatment is possible. One may show that there exist rings $R^{\text{univ}}(\overline{\rho})$, $R^{\text{loc}}(\overline{\rho})$ and $R^{k,\text{cris}}(\overline{\rho})$ whose spectrums are moduli spaces (in a very specific sense we shall not attempt to make precise here) for the following problems:

- The space $\text{Spec } R^{\text{univ}}(\overline{\rho})$ parametrizes lifts $\rho_0 : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(A)$ of $\overline{\rho}$ which are unramified outside $p$;
- The space $\text{Spec } R^{\text{loc}}(\overline{\rho})$ parametrizes lifts $\rho_0 : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \text{GL}_2(A)$ of $\overline{\rho}|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$;
- The space $\text{Spec } R^{k,\text{cris}}(\overline{\rho})$ parametrizes lifts $\rho_0 : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \text{GL}_2(A)$ of $\overline{\rho}|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ which are crystalline and Hodge-Tate of weights $\{0, k - 1\}$.

There is a closed embedding $\text{Spec } R^{k,\text{cris}}(\overline{\rho}) \hookrightarrow \text{Spec } R^{\text{loc}}(\overline{\rho})$ as well as a finite map $\text{Spec } R^{\text{univ}}(\overline{\rho}) \to \text{Spec } R^{\text{loc}}(\overline{\rho})$; this can be made into a closed embedding after adding finitely many variables, but we will just ignore this subtlety and assume for simplicity that it is an embedding also, since the conceptual difference is not so large. We then have from the definition that

$$\text{Spec } R_k(\overline{\rho}) = \text{Spec } R^{k,\text{cris}}(\overline{\rho}) \times_{\text{Spec } R^{\text{loc}}(\overline{\rho})} \text{Spec } R^{\text{univ}}(\overline{\rho}).$$

The picture is roughly as follows:\(^4\) the scheme $\text{Spec } R^{k,\text{cris}}(\overline{\rho})$ is 1-dimensional over $\mathbb{Z}_p$, the scheme $\text{Spec } R^{\text{univ}}(\overline{\rho})$ is 2-dimensional over $\mathbb{Z}_p$ and $\text{Spec } R^{\text{loc}}(\overline{\rho})$ is 3-dimensional and smooth (unless $p = 2, 3$). The following picture describes the situation after inverting $p$.

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\(^3\)Say, if you have two different newforms of weight $k$ and level $N$ with the same residual representation mod $p$ but not mod $p^2$.

\(^4\)Really one needs to fix determinants here; this gets rid of a few fudge directions in the deformation rings.
We now want to show that $R_k(\rho) = T_k(\rho)$. Here is a very rough sketch of the patching argument of Taylor-Wiles-Kisin. Instead of considering only $\text{Spec} R^\text{univ}(\bar{\rho})$, one can consider the larger space $\text{Spec} R^\text{univ}(\bar{\rho})$ which parametrizes lifts unramified outside a prime $l$ (so that $R^\text{univ}(\bar{p}) = R^\text{univ}_l(\bar{p})$). There will be certain primes $l$ (the Taylor-Wiles primes) for which $\text{Spec} R^\text{univ}_l(\bar{p}) \to \text{Spec} R^\text{loc}(\bar{p})$ will still be a closed embedding. One can vary $l$ so that $\text{Spec} R^\text{univ}_l(\bar{p})$ fills out more and more of $\text{Spec} R^\text{loc}(\bar{p})$. On the other hand, we can do the same process for the $T_{l,k}(\rho)$, and there are maps $R_{l,k}(\bar{p}) \to T_{l,k}(\bar{p})$. Since all of this is happening inside the compact (in the correct sense) space $\text{Spec} R^\text{loc}(\bar{p})$, one can pass to a convergent subsequence and (this is called “patching”) obtain rings $R_{\infty,k}(\bar{p})$ and $T_{\infty,k}(\bar{p})$ with a surjection $R_{\infty,k}(\bar{p}) \twoheadrightarrow T_{\infty,k}(\bar{p})$. (Notice that this construction works despite there being no maps between $\text{Spec} R^\text{univ}_N$ as $l$ varies!) In fact one can also patch simultaneously the modular forms of weight $k$ and level $l$ to obtain a $T_{\infty,k}(\bar{p})$-module (and hence a $R_{\infty,k}(\bar{p})$-module) which we call $M_{\infty,k}(\bar{p})$. Now the smoothness which one gets at this infinite level implies that the support $\text{Supp} M_{\infty,k}(\bar{p})$ of the sheaf on $R_{\infty,k}(\bar{p})$ associated to $M_{\infty,k}(\bar{p})$ is a union of components. It turns out that one can recover $R_k(\bar{p})$ and $T_k(\bar{p})$ from $R_{\infty,k}(\bar{p})$ and $T_{\infty,k}(\bar{p})$ by taking coinvariants, so any point of $\text{Spec} R_k(\bar{p})$ lying on $\text{Supp} M_{\infty,k}(\bar{p})$ is actually modular.

So to show that $\rho$ is modular, one wants to show that $M_{\infty,k}(\bar{p})$ is supported on the component associated to it. The assumption that there exists $g$ with $\bar{p}_g \cong \bar{p}$ means we have a point...
on $\text{Spec} R_k(\bar{p})$ which comes from $\text{Spec} T_k(\bar{p})$, so at least $\text{Supp} M_{\infty,k}(\bar{p})$ contains some component. By adding more technical assumptions one then do more; for example, under certain assumptions $\text{Spec} R_k(\bar{p})$ only contains one component, so in that case modularity lifting is proved.

4. The Taylor-Wiles-Kisin method for $GL_1$

In this section we will make the discussion of section 3 explicit in a simpler situation, that of $GL_1$ and level 1. There are versions of the Fontaine-Mazur conjecture and of the moduli spaces in this case as well, and they are easier to describe. Our description will actually assume modularity in this case (which amounts to the Kronecker-Weber theorem), and so we will not really prove a theorem; nevertheless, it will serve to illustrate the ideas present in the previous section.

So take $\bar{p}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_1(\mathbb{F}_p) = \mathbb{F}_p^\times$ to be the trivial representation. In this case one can show using class field theory that we have

$$R^{\text{univ}}(\bar{p}) = \mathbb{Z}_p \left[ \left[ \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \otimes \mathbb{Z} \right] \right] \cong \mathbb{Z}_p \left[ \left[ (1 + p\mathbb{Z}) \right] \right] \cong \mathbb{Z}_p \left[ [X] \right],$$

and $\text{Spec} R^{\text{univ}}(\bar{p})$ is identified with the locus \{\text{Frob}_p = 1\} (or equivalently \{\text{Y} = 0\}) in $\text{Spec} R^{\text{loc}}(\bar{p})$.

On the other hand, we also have a subring $R_{k,\text{cris}}(\bar{p})$ parametrizing crystalline liftings of $\bar{p}$ of Hodge-Tate weight $k$; these are just products of unramified characters with $\chi_{\text{cyc}}^k$, so $\text{Spec} R_{k,\text{cris}}(\bar{p})$ is identified with the locus $X = \chi_{\text{cyc}}(1+p)^k$ in $\text{Spec} R^{\text{loc}}(\bar{p})$ (it is nonempty only if $k$ is divisible by $p - 1$). If nonempty, the intersection $\text{Spec} R_k(\bar{p})$ is the point corresponding to the character $\chi^k: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_1(\mathbb{F}_p) = \mathbb{F}_p^\times$. This is described in the following picture.
What does Taylor-Wiles-Kisin patching do in this case? By adding a prime $l$, the ring $R_{\text{univ}}(\bar{\rho})$ is replaced by

$$R_{\text{univ}}^l(\bar{\rho}) = \mathbb{Z}_p[[\Delta_l \times (1 + p\mathbb{Z}_p)]] \cong \mathbb{Z}_p[\Delta_l][[X]],$$

where $\Delta_l = \mathbb{F}_l^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$. The map

$$\mathbb{Z}_p[[X,Y]] \cong R_{\text{loc}}^l(\bar{\rho}) \to R_{\text{univ}}^l(\bar{\rho}) \cong \mathbb{Z}_p[\Delta_l][[X]]$$

will now be given by mapping

$$Y \mapsto (p^{-1} \mod l \otimes 1) \in \Delta_l.$$

This will not always be an embedding on specs! To make it so, one needs to choose $l$ carefully. First, one better choose $l \equiv 1 \mod p$, otherwise $\Delta_l$ is trivial, $R_{\text{univ}}^l(\bar{\rho}) = R_{\text{univ}}^l(\bar{\rho})$ and we have not thickened up our original space. To ensure $\text{Spec}R_{\text{univ}}^l(\bar{\rho}) \to \text{Spec}R_{\text{loc}}^l(\bar{\rho})$ is still a closed embedding, it is enough to check the map on rings is surjective; since these rings are local and noetherian, it is enough to check the map on cotangent spaces is surjective, or that the map on tangent spaces is injective. This map can be described as the map on homs

$$\text{Hom}(\Delta_l \times (1 + p\mathbb{Z}_p), \mathbb{F}_p) \to \text{Hom}(\text{Frob}_p^* \times (1 + p\mathbb{Z}_p), \mathbb{F}_p)$$

which is induced by mapping $\text{Frob}_p$ to $(p^{-1} \mod l \otimes 1) \in \Delta_l$. In particular, we can see it is injective exactly when $p^{-1}$ generates $\Delta_l$, or what amount to the same, when $p$ is not a $p$-power mod $l$. 
Concluding, a Taylor-Wiles prime is now seen to be the same as a prime \( l \) satisfying the two conditions

\[
l \equiv 1 \mod p^n \text{ and } X^p = p \text{ has no solution } \mod l.
\]

To show the existence of such primes, we claim that any prime \( l \) which splits in \( \mathbb{Q}(\zeta_{p^n}) \) but not in \( \mathbb{Q}(\zeta_{p^n}, p^{1/p}) \) will satisfy both conditions; there are many such primes (a set of positive Dirichlet density) because of the Cebotarev density theorem. To prove the claim, note first that a prime splits in \( \mathbb{Q}(\zeta_{p^n}) \) if and only if it satisfies \( l \equiv 1 \mod p^n \), so that the first condition is satisfied. Now if \( v \) is a prime of \( \mathbb{Q}(\zeta_p) \) lying above such an \( l \), we have by Dedekind’s theorem that \( X^p - p \) has a solution in \( \mathbb{F}_v \) if and only if \( v \) splits in \( \mathbb{Q}(\zeta_p, p^{1/p}) \). Since we are assuming \( l \) does not split in \( \mathbb{Q}(\zeta_p, p^{1/p}) \), \( v \) does not split either, so \( X^p - p \) has no solution in \( \mathbb{F}_v \). But \( l \) splits in \( \mathbb{Q}(\zeta_p) \), so \( \mathbb{F}_v \) is identified with \( \mathbb{F}_l \) and we see that the second condition is satisfied also.

Now for each Taylor-Wiles prime \( l \), set \( R_{l,k}(\overline{p}) = R_{l}^{\text{univ}}(\overline{p}) \otimes_{R_{l,\text{loc}}(\overline{p})} R_{k,\text{cris}}(\overline{p}) \). By the previous discussion, we have a surjection \( \rho_{\infty}^{R_{l,\text{cris}}(\overline{p})} = R_{k,\text{cris}}(\overline{p}) \) onto \( R_{l,k}(\overline{p}) \), given by the map \( \mathbb{Z}_p[[\text{Frob}_{\overline{p}}^2]] \to \mathbb{Z}_p[\Delta_l] \) which sends \( \text{Frob}_{\overline{p}} \) to \((p^{-1} \mod l \otimes 1)\).

On the other hand, the Hecke algebra of conductor \( l \) is \( T_{l,k}(\overline{p}) = \mathbb{Z}_p[\Delta_l] \) (this doesn’t use class field theory, but rather follows from the definitions), and for each \( l \) we have maps \( R_{l,k}(\overline{p}) \to T_{l,k}(\overline{p}) \). Now write \( \Delta_\infty = \mathbb{Z}_p \) and fix a surjection \( \Delta_\infty \to \Delta_l \) for each Taylor-Wiles prime \( l \) (this requires many noncanonical choices). Let us then define \( S_\infty = \mathbb{Z}_p[[\Delta_\infty]] \), so that given our choices we have surjections \( S_\infty \to T_{l,k}(\overline{p}) \) for each \( l \), identifying \( T_{l,k}(\overline{p}) \) with a quotient of \( S_\infty \) by an augmentation ideal, and in particular giving each \( T_{l,k}(\overline{p}) \) and \( S_\infty \)-module structure. Because \( S_\infty \) is smooth, one can choose homomorphisms \( S_\infty \to R_{\infty,k}(\overline{p}) \) (depending on \( l \)) which lift the surjections \( S_\infty \to T_{l,k}(\overline{p}) \).

For any \( N \geq 1 \), write \( \epsilon_N \) for the ideal \( (p^N, \Delta_\infty^N - 1) \) of \( S_\infty \). They give a basis of open neighborhoods of \( S_\infty \) as \( N \to \infty \). For each Taylor-Wiles prime \( l \) congruent to \( 1 \mod p^N \) and each \( N' \leq N \), we have a commutative diagram \( D_{l,N'} \) given by

\[
\begin{array}{ccc}
S_\infty / \epsilon_{N'} & \longrightarrow & R_{\infty,k}(\overline{p}) / \epsilon_{N'} \\
\downarrow & & \downarrow \\
T_{l,k}(\overline{p}) / \epsilon_{N'} & \longrightarrow & T_k(\overline{p}) / p^{N'}
\end{array}
\]

When \( N \) is fixed, each \( l \) gives an ordered set of \( N \) diagrams \( \{D_{l,1}, ..., D_{l,N}\} \), and there is an obvious notion of isomorphism between two ordered sets of diagrams \( \{D_{l_{1,1}}, ..., D_{l_{1,N}}\} \) and \( \{D_{l_{2,1}}, ..., D_{l_{2,N}}\} \). As all the rings in each diagram are finite, there are only finitely many possibilities for the ordered set \( \{D_{l_{1,1}}, ..., D_{l_{1,N}}\} \) up to isomorphism of ordered sets of diagrams as the prime \( l \) varies. Thus we may pass to an infinite subsequence of \( l \)’s such that the ordered sets \( \{D_{l_{1,1}}, ..., D_{l_{1,N}}\} \) are all isomorphic. Since this can be done for any \( N' \), by passing to a diagonal subsequence of these subsequences, we have now obtained the following: a sequence of Taylor-Wiles primes \( l_N \) such that if \( N' \leq N \), there is an isomorphism of diagrams \( D_{l_{N,N'}} \cong D_{l_{N',N'}} \).

\[5\text{We introduce the new notation of } R_{\infty,k}(\overline{p}) \text{ because in other situations } R_{\infty,k}(\overline{p}) \text{ will be a power series over } R_{k,\text{cris}}(\overline{p}) \text{ and not literally equal to it.}\]
This allows us to define an $S_\infty$-algebra
\[ T_{\infty,k}(\overline{p}) := \lim_{\leftarrow} T_{l_N,k}(\overline{p})/\mathfrak{c}_N, \]
(the transition maps existing given from the isomorphism $D_{l_N,N'} \cong D_{l_N',N'}$) as well as a surjection $R_{\infty,k}(\overline{p}) \to T_{\infty,k}(\overline{p})$ induced from the maps $R_{\infty,k}(\overline{p}) \to T_{l_N,k}(\overline{p})/\mathfrak{c}_N$, which respects the $S_\infty$-algebra structure of both modules.

The map $R_{\infty,k}(\overline{p}) \to T_{\infty,k}(\overline{p})$ is of course an isomorphism, as we can see since we have made everything explicit using class field theory. If we pretend we didn’t know this already the argument would work by thinking of $T_{\infty,k}(\overline{p})$ as a module over $R_{\infty,k}(\overline{p})$. Indeed, both of these rings have the same Krull dimension\(^6\), and $R_{\infty,k}(\overline{p})$ is smooth, so a surjection must be an isomorphism.\(^7\) Finally, taking coinvariants (i.e. modding out by the augmentation ideal of $S_\infty$) one deduces that the map $R_k(\overline{p}) \to T_k(\overline{p})$ already has to be an isomorphism.

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\(^6\)This equality is called “the numerical coincidence”. The dimension of $T_{\infty,k}(\overline{p})$ has to do with the number of Taylor-Wiles primes one is adding, while the dimension of $R_{\infty,k}(\overline{p})$ has to do with the dimension of $R_k,\text{cris}(\overline{p})$, and it is these two numbers which will be related more generally.

\(^7\)It is this argument which will generalize to a more general situation, but one would need to consider $S_\infty$-module theoretic invariants for an $R_{\infty,k}(\overline{p})$-module $M_{\infty,k}(\overline{p})$ instead of just the ring $T_{\infty,k}(\overline{p})$. 

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