MODULARITY LIFTING FOR $\mathrm{GL}_2(F)$

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Abstract. These are notes for a final talk in a modularity lifting seminar. Following §5 of [TG13], we give the details for the proof of a modularity lifting theorem for $\mathrm{GL}_2(F)$ using the Taylor–Wiles-Kisin method in a simplified setting. We then briefly sketch how to obtain the proof in the general case.

1. Introduction

Our set up is the following. Let F be a totally real number field, $p > 5$ a prime unramified in F. Fix a field of coefficients E which is a finite extension of $\mathbb{Q}_p$, with $F^{\mathrm{Gal}} \subset E$, ring of integers $\mathcal{O}$, uniformizer λ and residue field $\mathbb{F} = \mathcal{O}/\lambda$. Our goal is to explain the proof of the following theorem.

Theorem 1.1. *Let $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation satisfying the following conditions.*
*(i) ρ **is geometric:** $\rho|_{G_{F_v}}$ is crystalline for each $v|p$, and there is some $2 \le k$ so that for each v, $\rho|_{G_{F_v}}$ has Hodge-Tate weights $\mathrm{HT}\left(\rho|_{G_{F_v}}\right)_\sigma = \{0, k-1\}$ at each embedding $\sigma : F_v \hookrightarrow E$.*
*(ii) $\overline{\rho}$ **is modular:** The reduction $\overline{\rho}$ has $\overline{\rho} \equiv \overline{\rho}_0$ for ρ_0 modular and $\mathrm{HT}(\rho) = \mathrm{HT}(\rho_0)$.*
*(iii) ρ **satisfies the Fontaine-Laffaille condition:** $0 \le k-1 \le p-1$.*
*(iv) ρ **satisfes the Taylor-Wiles condition:** $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F(\zeta_p))}$ is absolutely irreducible.*
Then ρ is modular.

To simplify the exposition, we shall work in the following setting, where ρ satisfies additional conditions:
(v) $\det\rho = \det\rho_0$.
(vi) $[F : \mathbb{Q}]$ *is even.*
(vii) $\overline{\rho}|_{G_{F_v}}$ *is Schur for each* $v|p$.
(viii) ρ, ρ_0 *are unramified outside of* p.
Conditions $(v)-(viii)$ can be removed, as will be explained in §6. Throughout the argument we will try to make it clear where each condition is used.
Here is the strategy of the proof of Theorem 1.1. Let $\mathbb{T} := \mathbb{T}_m$ be the Hecke algebra of Hilbert modular forms of level 1, parallel weight k, localized at the ideal corresponding to $\overline{\rho}_0$, and let $R := R_{\overline{\rho}}$ be the universal deformation ring of $\overline{\rho}$, parametrizing deformations with crystalline condition at $v|p$, unramified condition outside p, and fixed determinant $\det = \det\rho_0$ (the ring R exists because of conditions (iv) and (vii), otherwise one needs to add some framing variables). Because one can associate Galois representations to Hilbert eigenforms, we have a surjective map $R \twoheadrightarrow \mathbb{T}$ which induces a closed embedding $\mathrm{Spec}\mathbb{T} \hookrightarrow \mathrm{Spec}R$. Our aim will

be to show this closed embedding is in fact surjective. Theorem 1.1 will follow, because the representation ρ corresponds to a point of $\mathrm{Spec}R$ (by use of conditions (v) and (viii)), and the modularity of ρ amounts to this point being in the image of $\mathrm{Spec}\mathbb{T} \to \mathrm{Spec}R$.

By the Jacquet-Langlands correspondence, we can choose a quaternion algebra D over F so that ρ_0 is associated to a quaternionic modular eigenform of level 1 and parallel weight k. In fact D can be taken to unramified at all finite primes and compact at all infinite places, by condition (vi). To show $\mathrm{Spec}\mathbb{T} \hookrightarrow \mathrm{Spec}R$ is surjective, we shall consider the $\mathbb{T}$-module S of modular forms on $D^\times$ of level 1, parallel weight k, coefficients $\mathcal{O}$ and localized at the ideal corresponding to $\overline{\rho}_0$. It is a faithful $\mathbb{T}$-module by construction of $\mathbb{T}$. Using the map $R \to \mathbb{T}$ we can also restrict S to a module on R. This yields the following diagram:

$$\begin{tikzcd} \mathrm{Spec}\mathbb{T} \arrow[r, hook] \arrow[d, equal] & \mathrm{Spec}R \\ \mathrm{Supp}_{\mathbb{T}}S \arrow[r, "\sim"] & \mathrm{Supp}_R S \arrow[u, hook] \end{tikzcd} .$$

Here the equality $\mathrm{Spec}\mathbb{T} = \mathrm{Supp}_{\mathbb{T}}S$ holds because S is faithful as $\mathbb{T}$-module. The isomorphism $\mathrm{Supp}_{\mathbb{T}}S \xrightarrow{\sim} \mathrm{Supp}_R S$ holds because, on the level of sheaves, S as an R-module corresponds to the pushforward of S as a $\mathbb{T}$-module, which for the closed embedding $\mathrm{Spec}\mathbb{T} \hookrightarrow \mathrm{Spec}R$ coincides with the extension by zero.

Thus, Theorem 1.1. reduces to the following statement.

Theorem 1.2. $\mathrm{Supp}_R(S) = \mathrm{Spec}R$.

2. The Galois side

We have the following deformation rings, all with a fixed determinant condition:

- $R = R_{\overline{\rho}}$, the universal deformation ring of $\overline{\rho}$, parametrizing deformations with a crystalline condition at $v|p$ and which are unramified outside p (which exists by virtue of conditions (iv) and (vii)).
- $R_{\overline{\rho}}^{\mathrm{univ}}$, the universal deformation ring of $\overline{\rho}$, parametrizing deformations unramified outside p (which exists by virtue of condition (iv)).
- $R_{\overline{\rho}_v}^{\mathrm{univ}}$ for each $v|p$, the universal deformation ring of $\overline{\rho}_v := \overline{\rho}|_{G_{F_v}}$ (which exists by virtue of condition (vii)).
- R_v^{loc} for each $v|p$, the Kisin deformation ring parametrizing deformations of $\overline{\rho}_v$ which are crystalline and for which $\mathrm{HT} = \mathrm{HT}\,(\rho_v)$ (which exists by virtue of condition (vii)).

We would like to imagine that $\mathrm{Spec}R$ is just the intersection $\mathrm{Spec}R_{\overline{\rho}}^{\mathrm{univ}} \cap \mathrm{Spec}\left(\widehat{\bigotimes}_{v|p} R_v^{\mathrm{loc}}\right)$ taken inside $\mathrm{Spec}\left(\widehat{\bigotimes}_{v|p} R_{\overline{\rho}_v}^{\mathrm{univ}}\right)$. Unfortunately this won't be quite true in general, because $\mathrm{Spec}R_{\overline{\rho}}^{\mathrm{univ}} \to \mathrm{Spec}\left(\widehat{\bigotimes}_{v|p} R_{\overline{\rho}_v}^{\mathrm{univ}}\right)$ is not always a closed immersion. Rather, this map is finite. To make it an injection, it is enough to add variables to $\mathrm{Spec}\left(\widehat{\bigotimes}_{v|p} R_{\overline{\rho}_v}^{\mathrm{univ}}\right)$ to make the map on tangent spaces injective.

To do this, set

$$h = \dim_{\mathbb{F}} \ker\left(\mathrm{Tan}_{\mathbb{F}}\left(\mathrm{Spec}R^{\mathrm{univ}}_{\overline{\rho}}\right) \to \mathrm{Tan}_{\mathbb{F}}\left(\mathrm{Spec}\left(\widehat{\bigotimes_{v|p}} R^{\mathrm{univ}}_{\overline{\rho}_v}\right)\right)\right)$$
$$= \dim_{\mathbb{F}} \mathrm{H}^1\left(G_{F,\{v:v|p\}}, \mathrm{ad}^0\overline{\rho}(1)\right).$$

After adding variables $x_1, ..., x_h$, we now obtain a fiber product diagram

$$\begin{tikzcd}
\mathrm{Spec}R \arrow[r, hook] \arrow[d, hook] & \mathrm{Spec}\left(\left(\widehat{\bigotimes_{v|p}} R^{\mathrm{loc}}_v\right)[[x_1, ..., x_h]]\right) \arrow[d, hook] \\
\mathrm{Spec}R^{\mathrm{univ}}_{\overline{\rho}} \arrow[r, hook] & \mathrm{Spec}\left(\left(\widehat{\bigotimes_{v|p}} R^{\mathrm{univ}}_{\overline{\rho}_v}\right)[[x_1, ..., x_h]]\right)
\end{tikzcd}.$$

This diagram is particularly appealing for the following reason. Using standard arguments, one can compute that the expect dimension (over $\mathcal{O}$) of $\mathrm{Spec}R^{\mathrm{univ}}_{\overline{\rho}}$ is $2\,[F:\mathbb{Q}]$, the dimension of $\mathrm{Spec}\left(\left(\widehat{\bigotimes_{v|p}} R^{\mathrm{univ}}_{\overline{\rho}_v}\right)[[x_1, ..., x_h]]\right)$ is $3[F:\mathbb{Q}]+h$ and the dimension of $\mathrm{Spec}\left(\left(\widehat{\bigotimes_{v|p}} R^{\mathrm{loc}}_v\right)[[x_1, ..., x_h]]\right)$ is $[F:\mathbb{Q}]+h$. This suggests that the dimension of $\mathrm{Spec}R$ over $\mathcal{O}$ is 0, so that R would be finite over $\mathcal{O}$ if it were indeed true that $\mathrm{Spec}\mathbb{T} \xrightarrow{\sim} \mathrm{Spec}R$, provided that $\mathrm{Spec}\left(\left(\widehat{\bigotimes_{v|p}} R^{\mathrm{loc}}_v\right)[[x_1, ..., x_h]]\right)$ and $\mathrm{Spec}R^{\mathrm{univ}}_{\overline{\rho}}$ intersect transversaly enough.

The way to work around this problem is to thicken up $\mathrm{Spec}R$ to $\mathrm{Spec}\left(\left(\widehat{\bigotimes_{v|p}} R^{\mathrm{loc}}_v\right)[[x_1, ..., x_h]]\right)$ by "adding $[F:\mathbb{Q}]+h$ variables". Formally, we do the following. Given a set of $r = [F:\mathbb{Q}]+h$ primes Q of F not above p, let R_Q be the universal deformation ring with all the conditions of R, except that arbitray ramification is allowed at primes of Q. Clearly, there is a natural surjection $R_Q \twoheadrightarrow R$. Set $R_\infty := \left(\widehat{\bigotimes_{v|p}} R^{\mathrm{loc}}_v\right)[[x_1, ..., x_h]]$. We are going to choose specific sets Q so that there is a surjection $R_\infty \twoheadrightarrow R_Q$. In the language of Spec's, we have closed embeddings $\mathrm{Spec}R \subset \mathrm{Spec}R_Q \subset \mathrm{Spec}R_\infty$, so that patching all the $R_Q's$ together will let us view R_∞ as "a ring in $[F:\mathbb{Q}]+h$ variables over R". The problem is that for a random set Q we are not going to have $R_\infty \twoheadrightarrow R_Q$, for the same reason that we need to add variables to $\left(\widehat{\bigotimes_{v|p}} R^{\mathrm{loc}}_v\right)$ in the first place: the map on tangent spaces may not be injective. However, there exist special sets of primes Q for which this will happen. We will choose these sets to be even more special so to make certain technical aspects easier later on. The precise definition is the following.

Definition 2.1. An N-Taylor-Wiles set Q_N is a set of $r = [F:\mathbb{Q}]+h$ primes $\{w_1, ..., w_r\}$ of F such that:

1. For $w \in Q_N$:

- $\mathbb{F}_w \equiv 1 \mod p^N$
- $\overline{\rho}(\mathrm{Frob}_w)$ has two distinct eigenvalues.

2. There is a (noncanonical) surjection $R_\infty \twoheadrightarrow R_Q$ (and therefore a closed embedding $\mathrm{Spec}R_Q \hookrightarrow \mathrm{Spec}R_\infty$).

Condition 1 is quite easy to arrange but condition 2 is more tricky. One can rewrite it in terms of tangent spaces as the condition that

$$\dim_{\mathbb{F}} \ker\left(\mathrm{Tan}_{\mathbb{F}}\left(\mathrm{Spec}R_{\overline{\rho}}^{\mathrm{univ}}\right) \to \mathrm{Tan}_{\mathbb{F}}\left(\mathrm{Spec}\left(\widehat{\bigotimes_{v|p}} R_{\overline{\rho}_v}^{\mathrm{univ}}\right)\right)\right) =$$

$$\dim_{\mathbb{F}} \ker\left(\mathrm{Tan}_{\mathbb{F}}\left(\mathrm{Spec}R_{\overline{\rho},\{Q\}}^{\mathrm{univ}}\right) \to \mathrm{Tan}_{\mathbb{F}}\left(\mathrm{Spec}\left(\widehat{\bigotimes_{v|p}} R_{\overline{\rho}_v}^{\mathrm{univ}}\right)\right)\right).$$

This interpertation translates to the language of Galois cohomology. An Euler characteristic computation and condition (iv) give rise to the following ([TG13, Proposition 5.9]).

Proposition 2.1. *For each N there exists an N Taylor-Wiles set Q_N.*

For each N we now have a ring R_{Q_N} with a canonical surjection $R_{Q_N} \twoheadrightarrow R$ and with a noncanonical surjection $R_\infty \twoheadrightarrow R_{Q_N}$. There is a way to recover R from R_{Q_N}, as we shall now explain.

For each w, set $\Delta_w = \mathbb{F}_w^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and for each Q_N, set $\Delta_{Q_N} = \prod_{w\in Q_N} \Delta_w$. Set $J_{Q_N} = \mathcal{O}\left[\Delta_{Q_N}\right]$. It admits the following interpretation. Because of (i) of 2.1, we are in a special situation where the universal framed deformation ring of $\overline{\rho}_w$ is of the form

$$R_{\overline{\rho}_w}^{\square} = \mathcal{O}\left[[x,y,z]\right]\left[\Delta_w\right]$$

Under this isomorphism, inertia has finite image classified by $\mathcal{O}\left[\Delta_w\right]$. This means that liftings of $\overline{\rho}_w$ only have ramification coming from their inertial type being nontrivial; and these inertial types are classified by the components of $\mathrm{Spec}R_{\overline{\rho}_w}^{\square}$, which are the same as $\mathrm{Spec}\mathcal{O}[\Delta_w]$. Putting these all together for $w \in Q_N$, we get a homomorphism $J_{Q_N} = \mathcal{O}[\Delta_{Q_N}] \to R_{Q_N}$, corresponding to

$$\rho \mapsto \prod_w \rho_w|_{I_w}.$$

Consider the trivial inertial type $\mathrm{Spec}\mathcal{O} \to \mathrm{Spec}J_{Q_N}$, obtained by quotienting out J_{Q_N} by its augmentation ideal I_{Q_N}. Since all the ramification in Q_N is coming from the inertial type being nontrivial, the fiber of $\mathrm{Spec}R_{Q_N} \to \mathrm{Spec}J_{Q_N}$ over this point is just $\mathrm{Spec}R$. So geometrically we have the fiber product diagram

$$\begin{array}{ccc} \mathrm{Spec}R & \longrightarrow & \mathrm{Spec}\mathcal{O} = \mathrm{Spec}J_{Q_N}/I_{Q_N} \\ \downarrow & & \downarrow \\ \mathrm{Spec}R_{Q_N} & \longrightarrow & \mathrm{Spec}J_{Q_N} \end{array},$$

and algebraically we get $R = R_{Q_N} \otimes_{J_{Q_N}} J_{Q_N}/I_{Q_N}$.

3. The automorphic side

Since this has been discussed in previous talks, we shall only give brief reminders in this section. On the automorphic side, we have corresponding spaces of (quaternionic) modular forms S_{Q_N}. They are defined in the same way as S, except that at each $w \in Q_N$ there is some extra level structure U_w such that $\Gamma_1(w) \subset U_w \subset \Gamma_0(w)$ is allowed at each $w \in Q_N$, contaning these elements whose determinant is 1 in $\Delta_w = \mathbb{F}_w^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

There are two different things we need to know about $S_{Q_N}$for what comes next. The first thing is that this algebra has two different module structures.

- S_{Q_N} is a $\mathbb{T}_{Q_N}$-module, where $\mathbb{T}_{Q_N}$ means the Hecke algebra away from p and Q_N together with the U_w operators for $w \in Q_N$.
- S_{Q_N} is a $J_{Q_N} = \mathcal{O}[\Delta_{Q_N}]$-module. This is understood in the following way. For each $w \in Q_N$, we choose a generator δ of Δ_w and lift it to $\mathbb{F}_w^\times$. Then, thinking of quaternionic modular forms as functions on double cosets $\mathrm{D}^\times(\mathbb{Q})\backslash\mathrm{D}^\times(\mathbb{A})/U$, we map a function $f(t)$ to $f(t\mathrm{Diag}(\delta,1))$. In particular we notice that f is invariant under this action exactly if f is of a level which contains U_w.

We then have a diagram

$$\begin{tikzcd} J_{Q_N} \arrow[r] \arrow[drr] & R_{Q_N} \arrow[r] & \mathbb{T}_{Q_N} \arrow[d] \\ & & \mathrm{End}\,(S_{Q_N}) \end{tikzcd} .$$

Here the diagonal arrow $J_{Q_N} \to \mathrm{End}\,(S_{Q_N})$ is "local" (it is understood in terms of Satake parameters), while the composed arrow $J_{Q_N} \to R_{Q_N} \to \mathbb{T}_{Q_N} \to \mathrm{End}\,(S_{Q_N})$ is "global" (it is understood in terms of traces of Frobenius elements of a global Galois representations). As a consequence of local-global compatibility, the diagram commutes. Using the second interpretation, we can deduce in particular that $S_{Q_N} \otimes_{J_{Q_N}} J_{Q_N}/I_{Q_N} \cong S$.

The second thing we need to know is that S_{Q_N} is free over J_{Q_N} . This can be done by writing out S_{Q_N} explicitly as spaces of functions on double cosets. The assumption that $[F:\mathbb{Q}]$ is even is also useful in this analysis to make these spaces as simple as possible, because that allows one to choose a quaternion algebra ramified exactly at the infinite primes.

4. The patching argument

We are now ready to perform the patching argument. Recall that $R_\infty = \left(\widehat{\bigotimes_{v|p}} R_v^{\mathrm{loc}}\right)[[x_1,...,x_h]]$ and $\dim_{\mathcal{O}} R_\infty = [F:\mathbb{Q}] + h =: r$. Define $J_\infty = \mathcal{O}\,[[y_1,...,y_r]]$. Here we think of $y_1,...,y_r$ as variables encoding these dimensions that will be added to R through patching to get R_∞.

For each set of N Taylor-Wiles set Q_N, choose:

- A surjection $R_\infty \twoheadrightarrow R_{Q_N}$, which exists by construction;
- A surjection $J_\infty \twoheadrightarrow J_{Q_N}$, which exists because $J_{Q_N} = \mathcal{O}\,[\Delta_{Q_N}] \cong \mathcal{O}\,[\prod_{i=1}^r \mathbb{Z}/p^{n_i}]$;

- A dotted map $J_\infty \to R_\infty$ which completes the diagram

$$\begin{tikzcd} J_\infty \arrow[r, two heads] \arrow[d, dotted] \arrow[dr] & J_{Q_N} \arrow[d] \\ R_\infty \arrow[r, two heads] & R_{Q_N} \end{tikzcd}.$$

  Such a map exists because J_∞ is smooth.

Set $b_N = \mathrm{Ker}\left(J_\infty \to J_{Q_N}\right)$ and $c_N = \left(p^N, b_N\right)$. Clearly, c_N is an open ideal of J_∞. Moreover, $c_N \subset \left(p^N, (1+y_1)^{p^N} - 1, ..., (1+y_r)^{p^N} - 1\right)$, so $c_N \xrightarrow{N\to\infty} 0$. For $M \geq N$, the module S_{Q_M} is free over $J_{Q_M} = J_\infty/b_M$, so $S_{M,N} := S_{Q_M}/c_N$ is free over $J_\infty/c_N = J_{Q_M}/c_N$.

Fixing M, for each $N \leq M$ we have a diagram $D_{M,N}$:

$$\begin{tikzcd} J_\infty/c_N \arrow[r] \arrow[dr] & R_\infty/c_N \arrow[r, two heads] \arrow[d] & R/c_N \arrow[d] \\ & \mathrm{End}_{\mathcal{O}}\left(S_{M,N}\right) \arrow[r] & \mathrm{End}_{\mathcal{O}}\left(S/p^N\right) \end{tikzcd}.$$

Each object in this diagram is an $\mathcal{O}$-module of finite cardinality. In fact they have cardinality bounded by some constant $K(N)$ depending only on N. We can consider more generally the category $\mathcal{D}_N$ of diagrams D of $\mathcal{O}$-modules of finite cardinality $\leq K(N)$ of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ & \bullet \arrow[r] & \bullet \end{tikzcd},$$

with the obvious notion of a morphism. So $D_{M,N}$ is an object of $\mathcal{D}_N$ for each pair (M, N) with $N \leq M$.

We can also define a category $\mathcal{F}_N$ whose objects are diagrams $F_N = (D_1 \leftarrow D_2 \leftarrow ... \leftarrow D_N)$ with $D_i \in \mathcal{D}_i$. Clearly, each D_i can take only finitely many possibilities up to isomorphism; therefore, each $\mathcal{F}_N$ has only finitely objects up to isomorphism. There are obvious functors $\mathcal{F}_N \to \mathcal{F}_{N'}$.

Fix N. For each M with $N \leq M$, we obtain a tuple $F_{M,N} = (D_{M,1} \leftarrow D_{M,2} \leftarrow ... \leftarrow D_{M,N}) \in \mathcal{F}_N$. Thus, for each N we obtain infinitely many tuples $F_{N,N}, F_{N+1,N}, F_{N+2,N}, ... \in \mathcal{F}_N$. But $\mathcal{F}_N$ has only finitely many objects up to isomorphism, so by a diagnolization argument we can find a subsequence of M_i's so that the $F_{M_i,N}$'s are all compatible under the functors $\mathcal{F}_N \to \mathcal{F}_{N'}$. This means that we have succesfully constructed a sequence of compatibile diagrams $D_1 \leftarrow D_2 \leftarrow ... \leftarrow D_N \leftarrow ...$ with $D_i = D_{M_i,i}$.

Taking the limit over these diagrams, we now get a limit diagram of the form

$$\begin{tikzcd} J_\infty \arrow[r] \arrow[dr] & R_\infty \arrow[r] \arrow[d] & R \arrow[d] \\ & \mathrm{End}\left(S_\infty\right) \arrow[r] & \mathrm{End}\left(S\right) \end{tikzcd}.$$

Furthermore, S_∞ is finite free over J_∞. If $I_\infty = (y_1, ..., y_r)$ then $S = S_\infty \otimes_{J_\infty} J_\infty/I_\infty$ and $R = R_\infty \otimes_{J_\infty} J_\infty/I_\infty$.

5. Finishing the proof

Recall we want to prove that $\mathrm{Supp}_R(S) = \mathrm{Spec} R$. We prove the following.

Theorem 5.1. $\mathrm{Supp}_{R_\infty}(S_\infty) = \mathrm{Spec} R_\infty$.

It is enough to prove Theorem 5.1, because taking supports commutes with base change.

Lemma 5.1. *Let* (R, m) *be a noetherian local ring. Then* $\mathrm{depth}_R(M) \leq \dim \mathrm{Supp}_R(M)$.

This is proved by induction on depth.

Proof of Theorem 5.1. Recall that $J_\infty = \mathcal{O}[[y_1, ..., y_r]]$. By assumption, ρ_0 is modular, so S localized at the corresponding ideal is nonzero, so $\mathrm{Supp}_R(S) \neq \emptyset$. It follows that $\mathrm{Supp}_{R_\infty}(S_\infty) \neq \emptyset$. Choose a prime P in $\mathrm{Supp}_{R_\infty}(S_\infty)$ with $\dim R_\infty/P = \dim \mathrm{Supp}_{R_\infty}(S_\infty)$ (i.e. choose a component of $\mathrm{Supp}_{R_\infty}(S_\infty)$). We then have

$$r+1 = \dim J_\infty = \mathrm{depth}_{J_\infty}(S_\infty) \leq \mathrm{depth}_{R_\infty}(S_\infty) \leq \dim \mathrm{Supp}_{R_\infty}(S_\infty) = \dim R_\infty/P \leq \dim R_\infty = r+1.$$

Here, the second equality follows from the Auslander-Buchsbaum theory, and the second inequality follows from the lemma.

We see that all the inequalities are equalities. In particular, $\dim R_\infty/P = \dim R_\infty$, so P is a minimal prime. But condition (iii) implies that R_∞ is a domain, so $P = 0$. This finishes the proof, because $\mathrm{Supp}_{R_\infty}(S_\infty) \supset V(P) = V(0) = \mathrm{Spec} R_\infty$.

Remark 5.1. What steps in the above argument requires us to work with R_∞ and S_∞ instead of R and S? Why did we go through the trouble of performing patching? The point is that R_∞ is much better behaved than R:

- A priori we do not know R is finite over $\mathcal{O}$, so the step $\dim R_\infty = r + 1$ cannot be replaced for R.
- R_∞ is a domain but R is probably not.

In general, it seems the geometry of R_∞ is much simpler and easier to understand than that of R.

6. Removing unnecessary conditions

In this section, we briefly sketch how to get rid of conditions (v)-(viii) in the proof of the theorem.

6.1. **Base change.** To get rid of conditions (v)-(vi), one uses cyclic base change. If E is a totally real extension of F,and π is a cuspidal automorphic representation of $\mathrm{GL}_2(F)$, then a theorem of Langlands produces a cuspidal automorphic representation $\mathrm{BC}_{E/F}(\pi)$ of $\mathrm{GL}_2(E)$ for which $\rho_{\mathrm{BC}_{E/F}(\pi)} = \rho_\pi|_{G_E}$. In other words, this operation of base change gives an automorphic counterpart to the obvious operation of restriction of Galois group on the Galois side. Because the existence of this operation, a representation ρ of G_F will be modular if and only if $\rho|_{G_E}$ is modular.

This is very useful because of the following lemma due to Taylor:

Lemma 6.1. *Let F be a number field, and let S be a finite of places of F. For each $v \in S$, let L_v be a finite Galois extension of F_v. Then there is a finite solvable Galois extension E/F such that for each place w of E above a place $v \in S$ there is an isomorphism $L_v \cong E_w$ of F_v-algebras.*

If we take S to include all the infinite places, we may also ensure that if F is totally real then E is also. Performing such a base change doesn't ruin conditions (i)-(iv), so this allows us to perform many reductions. Using such a cyclic base change one reduces to the case where conditions (v) and (vi) are satisfied.

6.2. **Framed deformations.** Condition (vii) can be avoided by use of framed deformation rings instead of usual deformation rings. Essentially, the rings R_v^{loc} and $R_{\bar{\rho}_v}^{\mathrm{univ}}$ need to be replaced with framed versions where 3 variables are added for each such ring (this corresponds to the choice of a basis for the universal lifting). In addition the ring R needs to be slightly changed to account for these framing. Essentially not much is changed in the proofs but the numerics appearing in §2 and §5 become less pleasant, which is why we decided to add this condition to the simplifed discussion.

6.3. **Ihara avoidance.** The most serious restriction is given in condition (viii). To remove it, one needs to add to R the possibility of ramifiying at certain places away from p. Using the reduction with cyclic base change explained above, one may assume the inertial type of ρ_v at $v \nmid p$; the only ramification away from p one cannot get rid of is that at which the representation is semistable. This leads to a change in the definition of R_∞, by defining R_v^{loc} at v away from p, accounting for this ramification as well. The problem is now that R_∞ will no longer be an integral domain (it will have several components), and this was used crucially in argument of §5. Fortunately, there is an argument of Taylor ("Ihara avoidance") which bypasses this problem. Essentially, at each such $v \nmid p$, one defines a ring $R_v^{\mathrm{loc},'}$ which is very close related to R_v^{loc}. This leads to analogous definitions of R'_∞ and S'_∞. These are very similar to $R_v^{\mathrm{loc}}, R_\infty$ and S_∞ : in particular, their special fiber is the same. The upshot here is that unlike R_∞, the ring R'_∞ is an integral domain. This is useful because then the argument of §5 shows that at least $\mathrm{Supp}_{R'_\infty}(S'_\infty) = \mathrm{Spec} R'_\infty$. Since taking supports commutes with base change, one deduces $\mathrm{Supp}_{R_\infty}(S_\infty) = \mathrm{Spec} R_\infty$ and hence $\mathrm{Supp}_R(S) = \mathrm{Spec} R$.

References

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