MODULARITY LIFTING FOR GL₂(F)

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Abstract. These are notes for a final talk in a modularity lifting seminar. Following §5 of [TG13], we give the details for the proof of a modularity lifting theorem for GL₂(F) using the Taylor–Wiles-Kisin method in a simplified setting. We then briefly sketch how to obtain the proof in the general case.

1. Introduction

Our set up is the following. Let F be a totally real number field, p > 5 a prime unramified in F. Fix a field of coefficients E which is a finite extension of Q_p, with F^Gal ⊂ E, ring of integers O, uniformizer λ and residue field F = O/λ. Our goal is to explain the proof of the following theorem.

Theorem 1.1. Let ρ : GF → GL₂(O) be a continuous representation satisfying the following conditions.

(i) ρ is geometric: ρ|GF_v is crystalline for each v|p, and there is some 2 ≤ k so that for each v, ρ|GF_v has Hodge-Tate weights HT (ρ|GF_v)σ = {0, k − 1} at each embedding σ : F_v → E.

(ii) ρ is modular: The reduction p has p ≡ p₀ for ρ₀ modular and HT (ρ) = HT (ρ₀).

(iii) ρ satisfies the Fontaine-Laffaille condition: 0 ≤ k − 1 ≤ p − 1.

(iv) ρ satisfies the Taylor-Wiles condition: p|Gal(F/F(ζ_p)) is absolutely irreducible.

Then ρ is modular.

To simplify the exposition, we shall work in the following setting, where ρ satisfies additional conditions:

(v) det ρ = det ρ₀.

(vi) [F : Q] is even.

(vii) p|GF_v is Schur for each v|p.

(viii) ρ, ρ₀ are unramified outside of p.

Conditions (v) − (viii) can be removed, as will be explained in §6. Throughout the argument we will try to make it clear where each condition is used.

Here is the strategy of the proof of Theorem 1.1. Let T := T_m be the Hecke algebra of Hilbert modular forms of level 1, parallel weight k, localized at the ideal corresponding to p₀, and let R := R_p be the universal deformation ring of p, parametrizing deformations with crystalline condition at v|p, unramified condition outside p, and fixed determinant det = det ρ₀ (the ring R exists because of conditions (iv) and (vii), otherwise one needs to add some framing variables). Because one can associate Galois representations to Hilbert eigenforms, we have a surjective map R → T which induces a closed embedding Spec T → Spec R. Our aim will
be to show this closed embedding is in fact surjective. Theorem 1.1 will follow, because the representation \( \rho \) corresponds to a point of \( \text{Spec} R \) (by use of conditions (v) and (viii)), and the modularity of \( \rho \) amounts to this point being in the image of \( \text{Spec} \mathbb{T} \to \text{Spec} R \).

By the Jacquet-Langlands correspondence, we can choose a quaternion algebra \( D \) over \( F \) so that \( \rho_0 \) is associated to a quaternionic modular eigenform of level 1 and parallel weight \( k \). In fact \( D \) can be taken to be unramified at all finite primes and compact at all infinite places, by condition (vi). To show \( \text{Spec} \mathbb{T} \hookrightarrow \text{Spec} R \) is surjective, we shall consider the \( \mathbb{T} \)-module \( S \) of modular forms on \( D \times \) of level 1, parallel weight \( k \), coefficients \( \mathcal{O} \) and localized at the ideal corresponding to \( \overline{\rho}_0 \). It is a faithful \( \mathbb{T} \)-module by construction of \( \mathbb{T} \). Using the map \( R \to \mathbb{T} \) we can also restrict \( S \) to a module on \( R \). This yields the following diagram:

\[
\begin{array}{ccc}
\text{Spec} \mathbb{T} & \longrightarrow & \text{Spec} R \\
\downarrow & & \downarrow \\
\text{Supp}_T S & \sim & \text{Supp}_R S
\end{array}
\]

Here the equality \( \text{Spec} \mathbb{T} = \text{Supp}_T S \) holds because \( S \) is faithful as \( \mathbb{T} \)-module. The isomorphism \( \text{Supp}_T S \simarrow \text{Supp}_R S \) holds because, on the level of sheaves, \( S \) as an \( R \)-module corresponds to the pushforward of \( S \) as a \( \mathbb{T} \)-module, which for the closed embedding \( \text{Spec} \mathbb{T} \hookrightarrow \text{Spec} R \) coincides with the extension by zero.

Thus, Theorem 1.1. reduces to the following statement.

**Theorem 1.2.** \( \text{Supp}_R(S) = \text{Spec} R \).

2. **The Galois side**

We have the following deformation rings, all with a fixed determinant condition:

- \( R = R_{\overline{\rho}} \), the universal deformation ring of \( \overline{\rho} \), parametrizing deformations with a crystalline condition at \( v \mid p \) and which are unramified outside \( p \) (which exists by virtue of conditions (iv) and (vii)).
- \( R_{\overline{\rho}}^{\text{univ}} \), the universal deformation ring of \( \overline{\rho} \), parametrizing deformations unramified outside \( p \) (which exists by virtue of condition (iv)).
- \( R_{\overline{\rho}_v}^{\text{univ}} \) for each \( v \mid p \), the universal deformation ring of \( \overline{\rho}_v := \overline{\rho}|_{G_{F_v}} \) (which exists by virtue of condition (vii)).
- \( R_{\overline{\rho}_v}^{\text{loc}} \) for each \( v \mid p \), the Kisin deformation ring parametrizing deformations of \( \overline{\rho}_v \) which are crystalline and for which \( HT = HT (\overline{\rho}_v) \) (which exists by virtue of condition (vii)).

We would like to imagine that \( \text{Spec} R \) is just the intersection \( \text{Spec} R_{\overline{\rho}}^{\text{univ}} \cap \text{Spec} \left( \bigotimes_{v \mid p} R_{\overline{\rho}_v}^{\text{loc}} \right) \) taken inside \( \text{Spec} \left( \bigotimes_{v \mid p} R_{\overline{\rho}_v}^{\text{univ}} \right) \). Unfortunately this won’t be quite true in general, because \( \text{Spec} R_{\overline{\rho}}^{\text{univ}} \to \text{Spec} \left( \bigotimes_{v \mid p} R_{\overline{\rho}_v}^{\text{univ}} \right) \) is not always a closed immersion. Rather, this map is finite.

To make it an injection, it is enough to add variables to \( \text{Spec} \left( \bigotimes_{v \mid p} R_{\overline{\rho}_v}^{\text{univ}} \right) \) to make the map on tangent spaces injective.
definitions.

To do this, set
\[ h = \dim_{\mathbb{F}} \ker \left( \text{Tan}_{\mathbb{F}} \left( \text{Spec} R_{\mathbb{F}}^{\text{univ}} \right) \to \text{Tan}_{\mathbb{F}} \left( \text{Spec} \left( \hat{\otimes}_{\mathfrak{p}} R_{\mathfrak{p}}^{\text{univ}} \right) \right) \right) \]
\[ = \dim_{\mathbb{F}} H^1 \left( G_{F,v|v(p)}, \text{ad}^0 \rho(1) \right). \]

After adding variables \( x_1, \ldots, x_h \), we now obtain a fiber product diagram
\[
\begin{array}{ccc}
\text{Spec} R' & \to & \text{Spec} \left( \left( \hat{\otimes}_{\mathfrak{p}} R_{\mathfrak{p}}^{\text{univ}} \right) \left[ [x_1, \ldots, x_h] \right] \right) \\
\downarrow & & \downarrow \\
\text{Spec} R_{\mathbb{F}}^{\text{univ}} & \to & \text{Spec} \left( \left( \hat{\otimes}_{\mathfrak{p}} R_{\mathfrak{p}}^{\text{univ}} \right) \left[ [x_1, \ldots, x_h] \right] \right)
\end{array}
\]

This diagram is particularly appealing for the following reason. Using standard arguments, one can compute that the expectation dimension (over \( O \)) of \( \text{Spec} R_{\mathbb{F}}^{\text{univ}} \) is \( 2 [F : \mathbb{Q}] \), the dimension of \( \text{Spec} \left( \left( \hat{\otimes}_{\mathfrak{p}} R_{\mathfrak{p}}^{\text{univ}} \right) \left[ [x_1, \ldots, x_h] \right] \right) \) is \( 3[F : \mathbb{Q}] + h \) and the dimension of \( \text{Spec} \left( \left( \hat{\otimes}_{\mathfrak{p}} R_{\mathfrak{p}}^{\text{loc}} \right) \left[ [x_1, \ldots, x_h] \right] \right) \) is \( [F : \mathbb{Q}] + h \). This suggests that the dimension of \( \text{Spec} R \) over \( O \) is 0, so that \( R \) would be finite over \( O \) if it were indeed true that \( \text{Spec} \mathbb{T} \sim \to \text{Spec} R \), provided that \( \text{Spec} \left( \left( \hat{\otimes}_{\mathfrak{p}} R_{\mathfrak{p}}^{\text{loc}} \right) \left[ [x_1, \ldots, x_h] \right] \right) \) and \( \text{Spec} R_{\mathbb{F}}^{\text{univ}} \) intersect transversely enough.

The way to work around this problem is to thicken up \( \text{Spec} R \) to \( \text{Spec} \left( \left( \hat{\otimes}_{\mathfrak{p}} R_{\mathfrak{p}}^{\text{loc}} \right) \left[ [x_1, \ldots, x_h] \right] \right) \) by “adding \([F : \mathbb{Q}] + h\) variables”. Formally, we do the following. Given a set of \( r = [F : \mathbb{Q}] + h \) primes \( Q \) of \( F \) not above \( p \), let \( R_Q \) be the universal deformation ring with all the conditions of \( R \), except that arbitrary ramification is allowed at primes of \( Q \). Clearly, there is a natural surjection \( R_Q \to R \). Set \( R_{\infty} \) as \( \left( \hat{\otimes}_{\mathfrak{p}} R_{\mathfrak{p}}^{\text{loc}} \right) \left[ [x_1, \ldots, x_h] \right] \). We are going to choose specific sets \( Q \) so that there is a surjection \( R_{\infty} \to R_Q \). In the language of Spec’s, we have closed embeddings \( \text{Spec} R \subset \text{Spec} R_Q \subset \text{Spec} R_{\infty} \), so that patching all the \( R_Q \)s together will let us view \( R_{\infty} \) as “a ring in \([F : \mathbb{Q}] + h\) variables over \( R \)”. The problem is that for a random set \( Q \) we are not going to have \( R_{\infty} \to R_Q \), for the same reason that we need to add variables to \( \left( \hat{\otimes}_{\mathfrak{p}} R_{\mathfrak{p}}^{\text{loc}} \right) \) in the first place: the map on tangent spaces may not be injective. However, there exist special sets of primes \( Q \) for which this will happen. We will choose these sets to be even more special so to make certain technical aspects easier later on. The precise definition is the following.

**Definition 2.1.** An \( N \)-Taylor-Wiles set \( Q_N \) is a set of \( r = [F : \mathbb{Q}] + h \) primes \( \{ w_1, \ldots, w_r \} \) of \( F \) such that:
1. For \( w \in Q_N \):
   * \( \mathbb{F}_w \equiv 1 \mod p^N \)
   * \( \bar{\rho}(\text{Frob}_w) \) has two distinct eigenvalues.
2. There is a (noncanonical) surjection $R_\infty \twoheadrightarrow R_Q$ (and therefore a closed embedding $\text{Spec} R_Q \hookrightarrow \text{Spec} R_\infty$).

Condition 1 is quite easy to arrange but condition 2 is more tricky. One can rewrite it in terms of tangent spaces as the condition that

$$\dim_F \ker \left( \text{Tan}_F (\text{Spec} R_{\text{univ}}^\text{univ}) \to \text{Tan}_F \left( \text{Spec} \left( \otimes_{v|p} R_{\text{univ}}^\text{univ} \right) \right) \right) = \dim_F \ker \left( \text{Tan}_F (\text{Spec} R_{\text{univ}}) \to \text{Tan}_F \left( \text{Spec} \left( \otimes_{v|p} R_{\text{univ}}^\text{univ} \right) \right) \right).$$

This interpretation translates to the language of Galois cohomology. An Euler characteristic computation and condition (iv) give rise to the following ([TG13, Proposition 5.9]).

**Proposition 2.1.** For each $N$ there exists an $N$ Taylor-Wiles set $Q_N$.

For each $N$ we now have a ring $R_{Q_N}$ with a canonical surjection $R_{Q_N} \twoheadrightarrow R$ and with a noncanonical surjection $R_\infty \twoheadrightarrow R_{Q_N}$. There is a way to recover $R$ from $R_{Q_N}$, as we shall now explain.

For each $w$, set $\Delta_w = F^x \otimes \mathbb{Z}_p$, and for each $Q_N$, set $\Delta_{Q_N} = \prod_{w \in Q_N} \Delta_w$. Set $J_{Q_N} = \mathcal{O} [\Delta_{Q_N}]$.

It admits the following interpretation. Because of (i) of 2.1, we are in a special situation where the universal framed deformation ring of $\rho_w$ is of the form

$$R_{\rho_w}^\square = \mathcal{O} [[x, y, z]] [\Delta_w]$$

Under this isomorphism, inertia has finite image classified by $\mathcal{O} [\Delta_w]$. This means that liftings of $\rho_w$ only have ramification coming from their inertial type being nontrivial; and these inertial types are classified by the components of $\text{Spec} R_{\rho_w}^\square$, which are the same as $\text{Spec} \mathcal{O} [\Delta_w]$.

Putting these all together for $w \in Q_N$, we get a homomorphism $J_{Q_N} = \mathcal{O} [\Delta_{Q_N}] \to R_{Q_N}$, corresponding to

$$\rho \mapsto \prod_w \rho_w|_{I_w}.$$

Consider the trivial inertial type $\text{Spec} \mathcal{O} \to \text{Spec} J_{Q_N}$, obtained by quotienting out $J_{Q_N}$ by its augmentation ideal $I_{Q_N}$. Since all the ramification in $Q_N$ is coming from the inertial type being nontrivial, the fiber of $\text{Spec} R_{Q_N} \to \text{Spec} J_{Q_N}$ over this point is just $\text{Spec} R$. So geometrically we have the fiber product diagram

$$\begin{array}{ccc}
\text{Spec} R & \twoheadrightarrow & \text{Spec} \mathcal{O} = \text{Spec} J_{Q_N} / I_{Q_N} \\
\downarrow & & \downarrow \\
\text{Spec} R_{Q_N} & \twoheadrightarrow & \text{Spec} J_{Q_N}
\end{array}$$

and algebraically we get $R = R_{Q_N} \otimes_{J_{Q_N}} J_{Q_N} / I_{Q_N}$. 
3. The Automorphic Side

Since this has been discussed in previous talks, we shall only give brief reminders in this section. On the automorphic side, we have corresponding spaces of (quaternionic) modular forms $S_{Q_N}$. They are defined in the same way as $S$, except that at each $w \in Q_N$ there is some extra level structure $U_w$ such that $\Gamma_1(w) \subset U_w \subset \Gamma_0(w)$ is allowed at each $w \in Q_N$, containing these elements whose determinant is 1 in $\Delta_w = F_w \otimes \mathbb{Z}_p$.

There are two different things we need to know about $S_{Q_N}$ for what comes next. The first thing is that this algebra has two different module structures.

- $S_{Q_N}$ is a $T_{Q_N}$-module, where $T_{Q_N}$ means the Hecke algebra away from $p$ and $Q_N$ together with the $U_w$ operators for $w \in Q_N$.
- $S_{Q_N}$ is a $J_{Q_N} = \mathcal{O}[\Delta_{Q_N}]$-module. This is understood in the following way. For each $w \in Q_N$, we choose a generator $\delta$ of $\Delta_w$ and lift it to $F_w$. Then, thinking of quaternionic modular forms as functions on double cosets $D \times (Q) \backslash D \times (A) / U$, we map a function $f(t)$ to $f(t \text{Diag}(\delta, 1))$. In particular we notice that $f$ is invariant under this action exactly if $f$ is of a level which contains $U_w$.

We then have a diagram

$$
\begin{array}{c}
J_{Q_N} \rightarrow R_{Q_N} \rightarrow T_{Q_N} \\
\downarrow \downarrow \downarrow \downarrow \\
\text{End} (S_{Q_N})
\end{array}
$$

Here the diagonal arrow $J_{Q_N} \rightarrow \text{End} (S_{Q_N})$ is “local” (it is understood in terms of Satake parameters), while the composed arrow $J_{Q_N} \rightarrow R_{Q_N} \rightarrow T_{Q_N} \rightarrow \text{End} (S_{Q_N})$ is “global” (it is understood in terms of traces of Frobenius elements of a global Galois representations). As a consequence of local-global compatibility, the diagram commutes. Using the second interpretation, we can deduce in particular that $S_{Q_N} \otimes_{J_{Q_N}} J_{Q_N} / I_{Q_N} \cong S$.

The second thing we need to know is that $S_{Q_N}$ is free over $J_{Q_N}$. This can be done by writing out $S_{Q_N}$ explicitly as spaces of functions on double cosets. The assumption that $[F : \mathbb{Q}]$ is even is also useful in this analysis to make these spaces as simple as possible, because that allows one to choose a quaternion algebra ramified exactly at the infinite primes.

4. The Patching Argument

We are now ready to perform the patching argument. Recall that $R_\infty = \left( \bigotimes_{v|p} R_{v}^{\text{loc}} \right) [[x_1, \ldots, x_h]]$ and $\dim \mathcal{O} R_\infty = [F : \mathbb{Q}] + h =: r$. Define $J_\infty = \mathcal{O} [[y_1, \ldots, y_r]]$. Here we think of $y_1, \ldots, y_r$ as variables encoding these dimensions that will be added to $R$ through patching to get $R_\infty$.

For each set of $N$ Taylor-Wiles set $Q_N$, choose:

- A surjection $R_\infty \twoheadrightarrow R_{Q_N}$, which exists by construction;
- A surjection $J_\infty \twoheadrightarrow J_{Q_N}$, which exists because $J_{Q_N} = \mathcal{O}[\Delta_{Q_N}] \cong \mathcal{O} \prod_{i=1}^{r} \mathbb{Z}/p^{n_i}$. 

• A dotted map $J_\infty \to R_\infty$ which completes the diagram

$$
\begin{array}{c}
J_\infty \\
\vdots \\
R_\infty
\end{array} \rightarrow
\begin{array}{c}
J_Q \\
R_Q
\end{array}
$$

Such a map exists because $J_\infty$ is smooth.

Set $b_N = \text{Ker} (J_\infty \to J_{Q,N})$ and $c_N = \left( p^N, b_N \right)$. Clearly, $c_N$ is an open ideal of $J_\infty$. Moreover, $c_N \subseteq \left( p^N, (1 + y_1)^{p^N} - 1, \ldots, (1 + y_r)^{p^N} - 1 \right)$, so $c_N \xrightarrow{N \to \infty} 0$. For $M \geq N$, the module $S_{Q,M}$ is free over $J_{Q,M} = J_\infty / b_M$, so $S_{M,N} := S_{Q,M} / c_N$ is free over $J_\infty / c_N = J_{Q,M} / c_N$.

Fixing $M$, for each $N \leq M$ we have a diagram $D_{M,N}$:

$$
\begin{array}{c}
J_\infty / c_N \\
\downarrow \downarrow \\
R_\infty / c_N
\end{array} \rightarrow
\begin{array}{c}
R / c_N
\downarrow \downarrow \\
\text{End}_\mathcal{O} \left( S_{M,N} \right) \rightarrow \text{End}_\mathcal{O} \left( S / p^N \right)
\end{array}
$$

Each object in this diagram is an $\mathcal{O}$-module of finite cardinality. In fact they have cardinality bounded by some constant $K(N)$ depending only on $N$. We can consider more generally the category $\mathcal{D}_N$ of diagrams $D$ of $\mathcal{O}$-modules of finite cardinality $\leq K(N)$ of the form

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• \rightarrow • \rightarrow • \rightarrow ...
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with the obvious notion of a morphism. So $D_{M,N}$ is an object of $\mathcal{D}_N$ for each pair $(M, N)$ with $N \leq M$.

We can also define a category $\mathcal{F}_N$ whose objects are diagrams $F_N = (D_1 \leftarrow D_2 \leftarrow \ldots \leftarrow D_N)$ with $D_i \in \mathcal{D}_i$. Clearly, each $D_i$ can take only finitely many possibilities up to isomorphism; therefore, each $\mathcal{F}_N$ has only finitely objects up to isomorphism. There are obvious functors $\mathcal{F}_N \to \mathcal{F}_N'$.

Fix $N$. For each $M$ with $N \leq M$, we obtain a tuple $F_{M,N} = (D_{M,1} \leftarrow D_{M,2} \leftarrow \ldots \leftarrow D_{M,N}) \in \mathcal{F}_N$. Thus, for each $N$ we obtain infinitely many tuples $F_{N,N}, F_{N+1,N}, F_{N+2,N}, \ldots \in \mathcal{F}_N$. But $\mathcal{F}_N$ has only finitely many objects up to isomorphism, so by a diagonalization argument we can find a subsequence of $M_i$’s so that the $F_{M,N}$’s are all compatible under the functors $\mathcal{F}_N \to \mathcal{F}_{N'}$. This means that we have successfully constructed a sequence of compatible diagrams $D_1 \leftarrow D_2 \leftarrow \ldots \leftarrow D_N \leftarrow \ldots$ with $D_i = D_{M_i,i}$.

Taking the limit over these diagrams, we now get a limit diagram of the form

$$
\begin{array}{c}
J_\infty \\
\downarrow \downarrow \\
R_\infty
\end{array} \rightarrow
\begin{array}{c}
R
\downarrow \downarrow \\
\text{End} \left( S_\infty \right) \rightarrow \text{End} \left( S \right)
\end{array}
$$
Furthermore, \( S_\infty \) is finite free over \( J_\infty \). If \( I_\infty = (y_1, ..., y_r) \) then \( S = S_\infty \otimes_{J_\infty} J_\infty / I_\infty \) and 
\[ R = R_\infty \otimes_{J_\infty} J_\infty / I_\infty. \]

5. Finishing the proof

Recall we want to prove that \( \text{Supp}_R(S) = \text{Spec} R \). We prove the following.

**Theorem 5.1.** \( \text{Supp}_{R_\infty}(S_\infty) = \text{Spec} R_\infty. \)

It is enough to prove Theorem 5.1, because taking supports commutes with base change.

**Lemma 5.1.** Let \((R, m)\) be a noetherian local ring. Then \( \text{depth}_R(M) \leq \text{dim Supp}_R(M) \).

This is proved by induction on depth.

**Proof of Theorem 5.1.** Recall that \( J_\infty = \mathcal{O}[[y_1, ..., y_r]] \). By assumption, \( \rho_0 \) is modular, so \( S \) localized at the corresponding ideal is nonzero, so \( \text{Supp}_R(S) \neq \emptyset \). It follows that \( \text{Supp}_{R_\infty}(S_\infty) \neq \emptyset \). Choose a prime \( P \) in \( \text{Supp}_{R_\infty}(S_\infty) \) with \( \text{dim } R_\infty / P = \text{dim } \text{Supp}_{R_\infty}(S_\infty) \) (i.e. choose a component of \( \text{Supp}_{R_\infty}(S_\infty) \)). We then have 
\[ r+1 = \text{dim } J_\infty = \text{depth}_{J_\infty}(S_\infty) \leq \text{depth}_{R_\infty}(S_\infty) \leq \text{dim } \text{Supp}_{R_\infty}(S_\infty) = \text{dim } R_\infty / P \leq \text{dim } R_\infty = r+1. \]

Here, the second equality follows from the Auslander-Buchsbaum theory, and the second inequality follows from the lemma.

We see that all the inequalities are equalities. In particular, \( \text{dim } R_\infty / P = \text{dim } R_\infty \), so \( P \) is a minimal prime. But condition (iii) implies that \( R_\infty \) is a domain, so \( P = 0 \). This finishes the proof, because \( \text{Supp}_{R_\infty}(S_\infty) \supset V(P) = V(0) = \text{Spec } R_\infty. \)

**Remark 5.1.** What steps in the above argument requires us to work with \( R_\infty \) and \( S_\infty \) instead of \( R \) and \( S \)? Why did we go through the trouble of performing patching? The point is that \( R_\infty \) is much better behaved than \( R \):

- A priori we do not know \( R \) is finite over \( \mathcal{O} \), so the step \( \text{dim } R_\infty = r + 1 \) cannot be replaced for \( R \).
- \( R_\infty \) is a domain but \( R \) is probably not.

In general, it seems the geometry of \( R_\infty \) is much simpler and easier to understand than that of \( R \).

6. Removing unnecessary conditions

In this section, we briefly sketch how to get rid of conditions (v)-(viii) in the proof of the theorem.
6.1. **Base change.** To get rid of conditions (v)-(vi), one uses cyclic base change. If \( E \) is a totally real extension of \( F \), and \( \pi \) is a cuspidal automorphic representation of \( \text{GL}_2(F) \), then a theorem of Langlands produces a cuspidal automorphic representation \( \text{BC}_{E/F}(\pi) \) of \( \text{GL}_2(E) \) for which \( \rho_{\text{BC}_{E/F}(\pi)}|_{G_E} = \rho_{\pi}|_{G_E} \). In other words, this operation of base change gives an automorphic counterpart to the obvious operation of restriction of Galois group on the Galois side. Because the existence of this operation, a representation \( \rho \) of \( G_F \) will be modular if and only if \( \rho|_{G_E} \) is modular.

This is very useful because of the following lemma due to Taylor:

**Lemma 6.1.** Let \( F \) be a number field, and let \( S \) be a finite of places of \( F \). For each \( v \in S \), let \( L_v \) be a finite Galois extension of \( F_v \). Then there is a finite solvable Galois extension \( E/F \) such that for each place \( w \) of \( E \) above a place \( v \in S \) there is an isomorphism \( L_v \cong E_w \) of \( F_v \)-algebras.

If we take \( S \) to include all the infinite places, we may also ensure that if \( F \) is totally real then \( E \) is also. Performing such a base change doesn’t ruin conditions (i)-(iv), so this allows us to perform many reductions. Using such a cyclic base change one reduces to the case where conditions (v) and (vi) are satisfied.

6.2. **Framed deformations.** Condition (vii) can be avoided by use of framed deformation rings instead of usual deformation rings. Essentially, the rings \( R_{\text{loc}}^v \) and \( R_{\text{univ}}^{\rho_v} \) need to be replaced with framed versions where 3 variables are added for each such ring (this corresponds to the choice of a basis for the universal lifting). In addition the ring \( R \) needs to be slightly changed to account for these framing. Essentially not much is changed in the proofs but the numerics appearing in §2 and §5 become less pleasant, which is why we decided to add this condition to the simplified discussion.

6.3. **Ihara avoidance.** The most serious restriction is given in condition (viii). To remove it, one needs to add to \( R \) the possibility of ramifying at certain places away from \( p \). Using the reduction with cyclic base change explained above, one may assume the inertial type of \( \rho_v \) at \( v \nmid p \): the only ramification away from \( p \) one cannot get rid of is that at which the representation is semistable. This leads to a change in the definition of \( R_\infty \), by defining \( R_{\text{loc}}^v \) at \( v \) away from \( p \), accounting for this ramification as well. The problem is now that \( R_\infty \) will no longer be an integral domain (it will have several components), and this was used crucially in argument of §5. Fortunately, there is an argument of Taylor ("Ihara avoidance") which bypasses this problem. Essentially, at each such \( v \nmid p \), one defines a ring \( R_{\text{loc}}^v \) which is very close related to \( R_{\text{loc}}^v \). This leads to analogous definitions of \( R'_\infty \) and \( S'_\infty \). These are very similar to \( R_{\text{loc}}^v, R_\infty \) and \( S_\infty \) : in particular, their special fiber is the same. The upshot here is that unlike \( R_\infty \), the ring \( R'_\infty \) is an integral domain. This is useful because then the argument of §5 shows that at least \( \text{Supp}\, R'_\infty (S'_\infty) = \text{Spec}\, R'_\infty \). Since taking supports commutes with base change, one deduces \( \text{Supp}\, R_\infty (S_\infty) = \text{Spec}\, R_\infty \) and hence \( \text{Supp}\, R(S) = \text{Spec}\, R \).

**References**


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