LUBIN-TATE CHARACTERS ARE CRYSTALLINE

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Abstract. In this note I shall try to give a short and self contained proof that the Lubin-Tate characters are Crystalline, following ideas of Colmez and notes of Schneider.

1. Notations

Let $K/\mathbb{Q}_p$ be a finite extension with unramified extension K_0, uniformizer π. We fix a Frobenius power series $f(X) = \pi X + ... + X^q + ... \in \mathcal{O}_K[[X]]$, and a generator u_n of the Tate module of the associated formal group law. For an element $a \in \mathcal{O}_K^\times$ we have a corresponding Lubin-Tate power series $P_a(X) \in \mathcal{O}_K[[X]]$. We have a corresponding Lubin-Tate character $\chi : G_K = \mathrm{Gal}(\overline{K}/K) \to \mathcal{O}_K^\times$. The point of this note is to give a proof that χ is a crystalline representation.

2. Witt vectors and topology

Let $\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_p^\flat})$. We are going to need a relative version over K of this ring. Thus we let

$$\mathbf{A}_{\mathrm{inf},K} = \mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_K.$$

This is the "ramified witt vectors" from the book of Schneider. It has the following relevant properties, generalising those of $\mathbf{A}_{\mathrm{inf}}$:

1. There is a Teichmuller map $\mathcal{O}_{\mathbb{C}_p^\flat} \to \mathbf{A}_{\mathrm{inf},K}$, given simply by mapping $a \mapsto [a] \otimes 1$, where $[a]$ is the usual Teichmuller represenative for $\mathbf{A}_{\mathrm{inf}}$; but from now on we will just write $[a]$ for this element.

2. It has a Frobenius map, simply given by $F = \varphi^q \otimes 1$.

2. With the above notion of a Teichmuller map, every element of $\mathbf{A}_{\mathrm{inf},K}$ has a unique expansion

$$\alpha = \sum_{i=0}^{\infty} [\alpha_i] \pi^i.$$

3. $\mathbf{A}_{\mathrm{inf},K}$ has two canonical topologies. One of them is the π-adic topology; the other is the weak topology, which can be thought of as either the topology given by the isomorphism $\mathbf{A}_{\mathrm{inf}} \cong \left(\mathcal{O}_{\mathbb{C}_p^\flat}\right)^{\mathbb{N}}$, with the map given by the Teichmuller represenatives, and the topology on the latter being the product topology with the valuation on each $\mathcal{O}_{\mathbb{C}_p^\flat}$. Or, it can be thought of as the $([\pi^\flat], \pi)$-topology. [1] Note that it is harder for a sequence to converge in the π-adic

[1] The proof of this claim is Lemma 2.1.4 of Schneider, but it is not hard to prove it by hand once we note the image of multiplying by π^n is basically $0 \times 0 \times ... \times 0 \times \mathcal{O}_{\mathbb{C}_p^\flat} \times \mathcal{O}_{\mathbb{C}_p^\flat} \times ....$

topology, for the same reason it is the hardest for a sequence to converge in the discrete topology.

4. There is a homomorphism $\Theta : \mathbf{A}_{\mathrm{inf},K} \to \mathcal{O}_{\mathbb{C}_p}$, given by

$$\sum_{i=0}^{\infty}[\alpha_i]\pi^i \mapsto \sum_{i=0}^{\infty}\alpha_i^{\sharp}\pi^i,$$

whose kernel is spanned by $[\pi^\flat] - \pi$. Note that $\Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p}) = ([\pi^\flat], \pi)$, so the powers of $\Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$ also give the weak topology.

5. For one element $\alpha = \sum_{i=0}^{\infty}[\alpha_i]\pi^i$ to divide another element $\beta = \sum_{i=0}^{\infty}[\beta_i]\pi^i$ in $\mathbf{A}_{\mathrm{inf},K}$ with $[\alpha_0], [\beta_0] \neq 0$, it is necessary and sufficient that $|\alpha_0| \leq |\beta_0|$. In particular, for two such elements to differ by a unit it is necessary and sufficient that $|\alpha_0| = |\beta_0|$.

3. The $\{\}$ operator

Lemma 3.1. *1. An element $\alpha \in \mathbf{A}_{\mathrm{inf},K}$ lying in $\Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$ is topologically nilpotent for the $([\pi^\flat], \pi)$-topology.*
2. Let $\alpha \in \Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$. Then $(P_\pi \circ F^{-1})(\alpha) \in \Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$.

Proof. 1. is clear because $\Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p}) = ([\pi^\flat], \pi)$.
For 2., write $\alpha = \sum_{i=0}^{\infty}[\alpha_i]\pi^i$. Then we have

$$\left(P_\pi \circ F^{-1}\right)(\alpha) = P_\pi\Big(\sum_{i=0}^{\infty}[\alpha_i^{1/q}]\pi^i\Big) \equiv_\pi [\alpha_0^{1/q}]^q \equiv_\pi \alpha,$$

so in particular $(P_\pi \circ F^{-1})(\alpha) \equiv \alpha \bmod ([\pi^\flat], \pi)$, and hence is in $\Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$. □

The lemma shows that if $\alpha \in \mathbf{A}_{\mathrm{inf},K}$ is in $\Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$ then $(P_\pi \circ F^{-1})^i(\alpha)$ is defined for any $i \geq 0$.

The next Lemma is the key "Coleman Norm operator"-trick. The point is that $P_\pi \circ F^{-1}$ is a contracting operator, and hence has an eventual fixed point for any initial value.

Lemma 3.2. *Let $\alpha, \beta \in \Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$.*
1. $(P_\pi \circ F^{-1})(\alpha) \equiv \alpha \bmod \pi$.
2. If $\alpha = \beta \bmod \pi^i$ then $(P_\pi \circ F^{-1})(\alpha) = (P_\pi \circ F^{-1})(\beta) \bmod \pi^{i+1}$.
3. The limit $\lim_{i\to\infty}(P_\pi \circ F^{-1})^i(\alpha) =: \{\alpha\}$ exists.

Proof. Part 1 was already proved in Lemma 3.1. For part 2, we may write $\alpha = \beta + \pi^i\gamma$. Then

$$\begin{aligned}\left(P_\pi \circ F^{-1}\right)(\alpha) &= \left(P_\pi \circ F^{-1}\right)(\beta + \pi^i\gamma)\\ &= P_\pi(F^{-1}(\beta) + \pi^i F^{-1}(\gamma)),\end{aligned}$$

And since $P_\pi(X) \equiv_\pi X^q$, we have

$$P_\pi(F^{-1}(\beta) + \pi^i F^{-1}(\gamma)) \equiv_{\pi^{i+1}} \left(P_\pi \circ F^{-1}\right)(\beta).$$

Part 3 is now obvious from part 2. □

Proposition 3.1. *Let* $\alpha \in \mathbf{A}_{\mathrm{inf},K}$ *be an element of* $\Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$. *Then* $\{\alpha\}$ *is the unique element of* $\mathbf{A}_{\mathrm{inf},K}$ *satisfying*
1. $\alpha \equiv_\pi \{\alpha\}$, *and*
2. $F(\{\alpha\}) = P_\pi(\{\alpha\})$.

Proof. It is clear from Lemma 3.2 that $\{\alpha\}$ has both of these properties. Regarding the uniquenss, suppose β is an element satisfying both of this properties. Then $\{\alpha\} \equiv_\pi \beta$, and applying $(P_\pi \circ F^{-1})$ to both sides strengthens this congruence to an arbtirary level of precision by Lemma 3.2. □

From this uniquenss, it follows in particular that if $a \in \mathcal{O}_K$ then

$$P_a(\{\alpha\}) = \{P_a(\alpha)\}.$$

4. The elements ω, ξ and t

Let u_n be a generator of the Tate module of the formal group law, so

$$\overline{u} = (0 = \overline{u}_0, \overline{u}_1, \overline{u}_2, ...) \in \lim_{x^q \leftarrow x} \mathcal{O}_{\mathbb{C}_p}/\pi \cong \mathcal{O}_{\mathbb{C}_p}^\flat,$$

so $[\overline{u}] \in \mathbf{A}_{\mathrm{inf},K}$. It is tempting to think that $[\overline{u}]$ lies in $\ker\Theta$, but this is actually not true, as for example follows from the next proposition.

Proposition 4.1. *1.* $|\overline{u}^\sharp| = |\pi|^{\frac{q}{q-1}}$
2. $[\overline{u}] \in \Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$.

Proof. To prove 1, one must compute the norm of the limit $\lim_{n\to\infty} u_n^{q^n}$, and by Lubin Tate theory the norm of u_n is $|\pi|^{\frac{1}{q^{n-1}(q-1)}}$.
2 now follows immdiately from 1. □

In particular, part 2 of the proposition shows that $\{[u]\}$ is defined.
We let

$$\xi = \{[\overline{u}]\}/\{[\overline{u}^{1/q}]\}.$$

Proposition 4.2. *The element* ξ *is a generator of* $\ker\Theta$ *in* $\mathbf{A}_{\mathrm{inf},K}$.

Proof. Let's first verify that $\{[\overline{u}]\} \in \ker\Theta$. Write $P_\pi^{[n]}$ for $P_\pi \circ P_\pi \circ ... \circ P_\pi$. Note that $(P_\pi \circ F^{-1})^n([\overline{u}]) = P_\pi^{[n]}(F^{-n}([\overline{u}])$, because the action of F on the coefficients in $\mathcal{O}_K$ are trivial. We have

$$\Theta(\{[\overline{u}]\}) = \Theta\left(\lim_{n\to\infty} \left(P_\pi \circ F^{-1}\right)^n([\overline{u}])\right) = \Theta\left(\lim_{n\to\infty} P_\pi^{[n]}(F^{-n}([\overline{u}]))\right)$$

$$= \lim_{n\to\infty} P_\pi^{[n]}(\Theta(F^{-n}([\overline{u}]))) = \lim_{n\to\infty} P_\pi^{[n]}\left(\left(\overline{u}^{1/q^n}\right)^\sharp\right).$$

Now, $\overline{u}^{1/q^n} = (\overline{u}_n, \overline{u}_{n+1}, ...)$, so letting $v_n = \left(\overline{u}^{1/q^n}\right)^\sharp$ we have $v_n = u_n \bmod \pi$. Now as $P_\pi(X) \equiv_\pi X^q$, we have $0 = P_\pi^{[n]}(u_n) \equiv_{\pi^{n+1}} P_\pi^{[n]}(v_n)$. Thus

$$\Theta(\{[\overline{u}]\}) = \lim_{n\to\infty} P_\pi^{[n]}(v_n) = 0.$$

On the other hand, the same argument shows that $\Theta(\{[\overline{u}^{1/q}]\}) \equiv_{\pi^{n+1}} P_\pi^{[n]}(u_{n-1})$, which does not tend to 0. So the quotient is still in the kernel when we extend Θ to the fraction field.

To see it is a generator, we note that $[\pi^\flat] - \pi$ is a generator, and $|\left(\pi^\flat\right)^\sharp| = |\pi|$. On the other hand for ξ the norm of the first Teichmuller digit is $\left(|\overline{u}^\sharp|\right)^{1-\frac{1}{q}} = |\pi|$, by Proposition 4.1. Thus they 5 of section 2 they differ by a unit in $W(\mathcal{O}_{\mathbb{C}_p}^\flat)$, so ξ is a generator of $\ker\Theta$. $\square$

We now define two other elements. We let $\omega = \{[\overline{u}]\}$, so the $\omega = c\xi$ for $c = \{[\overline{u}^{1/q}]\}$. We let $t = \mathrm{Log}(\omega)$,where $\mathrm{Log} \in K[[X]]$ is the logarithm of the Lubin-Tate formal group law. Note that (for example by the proof in Iwasawa's book) $\mathrm{Log} = \sum_{n\geq 1} \frac{a_n}{n} X^n$ where $a_n \in \mathcal{O}_K$, so the coefficients are well behaved. At this point t is just some kind of formal object, but we will see this power series converges in a suitable sense.

We let

$$A^0_{\max,K} = \mathbf{A}_{\inf,K}\left[\frac{\xi^m}{\pi^m}\right],$$

and $A_{\max,K}$ be the π-adic completion of $A^0_{\max,K}$. We note that F extends to an element of this ring, as usual. We claim that

Proposition 4.3. *The series $Log(\omega)$ converges to an element $t \in A_{\max,K}$.*

Proof. Letting $\mathrm{Log} = \sum_{n\geq 1} \frac{a_n}{n} X^n$ for $a_n \in \mathcal{O}_K$ and $\omega = c\xi$, we have

$$\mathrm{Log}(\omega) = \sum_{n\geq 1} \frac{a_n}{n}\omega^n = \sum_{n\geq 1} \frac{a_n c^n \pi^n}{n}\left(\frac{\xi^n}{\pi^n}\right),$$

so this converges π-adically in $A_{\max,K}$. $\square$

Proposition 4.4. *We have*

$$F(t) = \pi t, g(t) = \chi(g)t.$$

Proof. By the end of section 3, we have $F(\omega) = P_\pi(\omega)$ and $g(\omega) = \{g([\overline{u}])\} = \{P_{\chi(g)}([\overline{u}])\} = P_{\chi(g)}(\omega)$. So the result is clear by definition of the logarithm. $\square$

In other words, we have shown that there is a Lubin-Tate period in $A_{\max,K}$.

5. Lubin-Tate characters are Crystalline

Lemma 5.1. *$B_{\max,K} = B_{\max} \otimes_{K_0} K$.*

Proof. This should follow from a uniformizer free definition of $A_{\max,K}$, I think in Colmez it is defined to be the completion of $p^{-1}\Theta^{-1}(\pi\mathcal{O}_{\mathbb{C}_p})$. Then one should be able to compare the two. Regardless the containment $\supset$ is not hard. $\square$

Theorem 5.1. *Lubin-Tate characters are crystalline.*

Proof. Let $\chi : G_K \to K^\times$ be the Lubin-Tate character, and let V_χ be the corresponding representation, thought of as a $\mathbb{Q}_p$ vector space of dimension $d = [K : \mathbb{Q}_p]$. Then it suffices to show $\left(V_\chi \otimes_{\mathbb{Q}_p} B_{\max}\right)^{G_K}$ has dimension d. Well, we have

$$V_\chi \otimes_{\mathbb{Q}_p} B_{\max} = V_\chi \otimes_K \left(K \otimes_{\mathbb{Q}_p} B_{\max}\right) = V_\chi \otimes_K (K \otimes_{K_0} K_0 \otimes_{\mathbb{Q}_p} B_{\max}).$$

The G_K-action on K_0 is trivial, so $K_0 \otimes_{\mathbb{Q}_p} B_{\max} \cong \oplus_{i=1}^f B_{\max}$ as $K_0[G_K]$-modules, where if $a \in K_0$, the action on the right hand side is given by

$$a(b_1, ..., b_f) = (\sigma_1(a)b_1, ..., \sigma_f(a)b_f),$$

where σ_i are the different embeddings of K_0 in $B_{\max}$. In particular, $K \otimes_{K_0} \oplus_{i=1}^f B_{\max} = \oplus_{i=1}^f B_{\max,K}$ (the embedding of K_0 in $B_{\max}$ can differ up to an automorphism in each component, but this does not matter). So

$$V_\chi \otimes_{\mathbb{Q}_p} B_{\max} = V_\chi \otimes_K \left(K \otimes_{K_0} \oplus_{i=1}^f B_{\max}\right) = \oplus_{i=1}^f V_\chi \otimes_K B_{\max,K}.$$

Now by proposition 4.4 we have

$$V_\chi \otimes t^{-1} \subset (V_\chi \otimes_K B_{\max,K})^{G_K},$$

and it is a 1-dimensional K-vector space, hence an $e = [K : K_0]$ dimensional K_0-vector space. So $\left(V_\chi \otimes_{\mathbb{Q}_p} B_{\max}\right)^{G_K}$ is an $ef = d$ dimensional K_0-vector space, as required. □