KÄHLER DIFFERENTIALS OF INSEPARABLE FIELD EXTENSIONS

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ABSTRACT. Let L/K be a purely inseparable field extension, and denote by $\Omega_{L/K}$ the Kähler differentials of L over K. We show that $\dim_L \Omega_{L/K}$ is equal to the minimal number of generators for L over K, and give a few examples.

1. Introduction

Let L/K be an extension of fields, and denote by $\Omega_{L/K}$ the Kähler differentials of L over K. It is well known that $\dim_L \Omega_{L/K} \geq \text{tr.deg}(L/K)$, with equality if and only if L/K is separable. Thus, in the separable case, the Kähler differentials are completely understood. On the other hand, if L/K is an inseparable extension, then $\dim_L \Omega_{L/K} > 0$. The purpose of this note is to give a description of this invariant in terms of the extension L/K, for which the author has not found any reference.

The interesting case is that of a purely inseparable extension. Let $m(L/K) \in \mathbb{N} \cup \{\infty\}$ be the minimal number of generators of L over K. We have the following

Theorem 1.1. *If L/K is purely inseparable,* $\dim_L \Omega_{L/K} = m(L/K)$.

Apart from giving a description for the invariant $\dim_L \Omega_{L/K}$ in terms of the extension, the theorem can be used to compute $m(L/K)$. We will illustrate this in section 3.

2. Proving the theorem

In this section, we suppose that L/K is a purely inseparable extension. As Kähler differentials are compatible with filtered colimits, the case $[L:K] = \infty$ in the Theorem follows from the case $[L:K] < \infty$, so we may and do suppose that L/K is finite.

Lemma 2.1. $m(L/K) = m(L/L^p K)$.

Proof. The inequality $m(L/K) \geq m(L/L^p K)$ is obvious. On the other hand, suppose $L = L^p K(\alpha_1, ..., \alpha_r)$. Arguing by induction, one sees that $L = L^{p^n} K(\alpha_1, ..., \alpha_r)$ for all $n \geq 1$. As L/K is purely inseparable, $L^{p^n} \subset K$ for n large enough, so $L = K(\alpha_1, ..., \alpha_r)$. □

With this Lemma in place, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $K' = L^p K$; we claim that $\dim_L \Omega_{L/K} = \dim_L \Omega_{L/K'}$. Indeed, the extensions $K \subset K' \subset L$ give the "first fundamental sequence":

$$\Omega_{K'/K} \otimes_{K'} L \to \Omega_{L/K} \to \Omega_{L/K'} \to 0.$$

The image of the leftmost map is spanned by the elements $dx \in \Omega_{L/K}$ with $x \in K' = L^p K$. These are all zero in $\Omega_{L/K}$, so the middle map is an isomorphism, and the claim is proved.

On the other hand, by Lemma 2.1, we have $m(L/K) = m(L/K')$. Thus it suffices to prove the theorem for K', so we reduce to the case where $L^p \subset K$.

Let $r = m(L/K)$, so $L = K(\alpha_1, ..., \alpha_r)$ with r minimal. As $L^p \subset K$, $\alpha_1^p, ..., \alpha_r^p \in K$, we have a surjection of K-algebras

$$\frac{K[X_1, ..., X_r]}{(X_1^p - \alpha_1^p, ..., X_r^p - \alpha_r^p)} \twoheadrightarrow L,$$

which must be an isomorphism by counting dimensions. But it is clear that the Kähler differentials of $\frac{K[X_1,...,X_r]}{(X_1^p-\alpha_1^p,...,X_r^p-\alpha_r^p)}$ over K have $dX_1, ..., dX_r$ as a basis. This concludes the proof. □

Remark 2.1. The argument in the proof of Theorem 1.1 shows that when $L^p \subset K$, one has $[L : K] = \dim_L \Omega_{L/K}$, and thus also $[L : K] = m(L/K)$. It then follows from Lemma 2.1 that in general $[L : L^pK] = m(L/K)$. This reproves Theorem 6 of [BeMa].

3. Examples

We give two simple examples of applications of the theorem.

Example 3.1. Let $K = \mathbb{F}_p(t, s)$ be the field of rational functions of two independent variables over $\mathbb{F}_p$. Consider its purely inseparable extension $L = \mathbb{F}_p\left(t^{1/p}, s^{1/p}\right)$. Its module of differentials over K is 2-dimensional, generated by $d\left(t^{1/p}\right)$ and $d\left(s^{1/p}\right)$, so the extension L/K is not simple.

Example 3.2. Let $K = \mathbb{F}_p(t)$ be the field of rational functions of one variable over $\mathbb{F}_p$. Consider its purely inseparable extension $L = \mathbb{F}_p\left(t^{1/p}, \left(t^{2/p} + 1\right)^{1/p}\right)$. We may write

$$L = \frac{K[X, Y]}{(X^p - t, Y^p - X^2 - 1)}.$$

Then $\Omega_{L/K}$ is generated by dX and dY, subject to the relation $2dX = 0$; it is 1-dimensional if $p \neq 2$ and 2 dimensional if $p = 2$. By Theorem 1.1, this extensions is simple if and only if $p \neq 2$.

References

[BeMa] Becker, M. F.; MacLane, S. The minimum number of generators for inseparable algebraic extensions. Bull. Amer. Math. Soc. 46 (1940), no. 2, 182--186. https://projecteuclid.org/euclid.bams/1183502442