GROUP PRESENTATIONS AND \(p\)-ADIC ANALYTIC GROUPS

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In this talk, I will first present the Golod-Shafarevich theorem about finite \(p\)-groups; then I will present a generalization of the theorem to \(p\)-adic analytic \(p\)-groups. Finally I will show some applications to presentations of discrete groups. This talk is based in full on the article “Group presentation, \(p\)-adic analytic groups and lattices in \(SL_2(\mathbb{C})\)” by Alex Lubotzky.

1 Notation and preliminaries

Recall that a pro-\(p\) group is an inverse limit of finite \(p\)-groups. When given a discrete group \(\Gamma\), taking the inverse limit \(\lim_{\leftarrow} \Gamma/N\) where \(N\) is a normal subgroup of index a finite power of \(p\) yields pro-\(p\) group called the pro-\(p\) completion of \(\Gamma\). We denote it by \(\Gamma_{\hat{p}}\). It is characterized by a universal property: every homomorphism from \(\Gamma\) to a pro-\(p\) group \(G\) factors uniquely through \(\Gamma_{\hat{p}}\) to a continuous homomorphism.

For a finitely generated group, \(d(G)\) denotes the minimal number of topological generators. The Frattini subgroup \(\Phi(G)\) is defined as the intersection of all the maximal open subgroups of \(G\). It is elementary to show that \(d(G) = d(G/\Phi(G))\).

If \(G\) is a pro-\(p\) group, then \(\Phi(G) = [G, G]/G^p\) so that \(d(G) = \dim_{\hat{F}_p}(G/\Phi(G))\).

A presentation of a finitely generated pro-\(p\) group \(G\) is a pair \(<X; R>\) where \(|X|\) is finite and \(\hat{F}_p/\langle R\rangle \simeq G\) for the pro-\(p\) completion of the free group on \(|X|\) generators, i.e. \(\hat{F}_p = (F[X])_{\hat{p}}\).

Finally, a pro-\(p\) group is said to be \(p\)-adic analytic if it has the structure of a manifold over \(Q_p\) that is compatible with the group structure. A finite \(p\)-group is \(p\)-adic analytic, and so is \(Z_p\).

2 The Golod-Shafarevich Theorem and its generalization

**Theorem (Golod-Shafarevich).** Let \(G\) be a finite \(p\)-group with minimal presentation \(<X; R>\) (in the usual sense). Then \(|R| \geq \frac{|X|^2}{4}\).

**Remark:** the inequality is in fact strict whenever \(G \neq 1\).
I will not present here the proof of this theorem, but we will discuss a certain portion of it when we talk about the generalization. The proof I know of mainly uses cohomology of groups.

Side note: the original motivation for this theorem is from number theory. It can be used to construct an infinite tower of Hilbert class field extensions, thus answering to the negative a conjecture of Furtwangler.

The following is a generalization of the previous theorem for $p$-adic analytic groups.

**Theorem (Lubotzky).** Let $G$ be a $p$-adic analytic pro-$p$ group different from the $p$-adic integers $\mathbb{Z}_p$. If $<X;R>$ is a minimal presentation of $G$ then

$$|R| \geq \frac{|X|^2}{4}$$

It can also be shown that that is inequality is strict except for three exceptional cases.

*(very) Partial proof*: we set some notation. Put $d = |X|, r = |R|$. Let $G = \lim_{\leftarrow} G_i$ where the $G_i$ are finite $p$-groups. Define $B = \mathbb{F}_p[[G]] = \lim_{\leftarrow} \mathbb{F}[G_i]$. Let $I(G) = \text{Ker}(B \xrightarrow{\text{sum}} \mathbb{F}_p)$ (the augmentation ideal) and $C_n(G) = \dim_{\mathbb{F}_p} B/I(G)^n$.

We will take it as a fact that if $\sum C_n(G)t^n$ is absolutely convergent for $0 \leq t < 1$ then $\varphi(t) = 1 - dt + rt^2 > 0$ for $0 \leq t < 1$. In the proof for the Golod-Shafarevich theorem, it is shown that the series $\sum C_n(G)t^n$ satisfies the convergence condition by commenting that $C_n(G) = |G|$ for large enough $n$.

By a result of Lazard, a finitely generated pro-$p$ group is $p$-adic analytic if and only if $C_n(G)$ has polynomial growth. Hence the convergence condition is still satisfied for a general $p$-adic analytic group, so we have $\varphi(t) = 1 - dt + rt^2 > 0$ for $0 \leq t < 1$ (the reader may now safely forget about the origin of this $\varphi(t)$).

Setting $t = 1$ we see that $1 - d + r \geq 0$, i.e. $r \geq d - 1$. If $r = 0, d = 0$ this is the case $\mathbb{Z}_p$. If $r = 0, d = 1$ this clearly satisfies the theorem. We can now assume that $r > 0$. Suppose by contradiction that $r < \frac{d^2}{4}$. Plugging $t_0 = \frac{d}{2r}$ into $\varphi(t)$ gives $t_0 \leq 0$, which is possible only if $t_0 \geq 1$, i.e. $d \geq 2r$. But we also have that $r \geq d - 1$, so we must have $d = 2, r = 1$. But in this case $r = \frac{d^2}{4}$, so this is a contradiction, and we're done.

### 3 Presentations of pro-$p$ completions

For the next theorem, we first prove two Lemmas.

**Lemma 1.** Let $G$ be a finitely generated pro-$p$ group and let $<X;R>$ be a presentation of $G$. Then there is a minimal presentation $<Y;S>$ of $G$ such that $|S| - |Y| = |R| - |X|$. (we can change a given presentation to a minimal presentation without changing the difference between the number of relations and generators).
Proof. Let \( F = \hat{F} < X > \) be the free pro-\( p \) group on the elements of \(|X|\). Then there is an epimorphism \( \varphi : F \rightarrow G \) with kernel \( \langle R \rangle \). This induces a map \( \overline{\varphi} : F/\Phi(F) \rightarrow G/\Phi(G) \) (it is easy to see that \( \varphi([F,F]^p) = [G,G]^p \), with or without closures). Let \( M/\Phi(F) \) be the kernel of \( \overline{\varphi} \). Notice that this is then a subspace of the \( F_p \) vector space \( F/\Phi(F) \).

I claim that this subspace is generated by the image of \( R \). To see this, consider the following commutative diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \Phi(F) & \longrightarrow & F & \longrightarrow & F/\Phi(F) & \longrightarrow & 1 \\
| & \downarrow \varphi | & \downarrow \overline{\varphi} & & \downarrow & & \downarrow & \text{1} \\
1 & \longrightarrow & \Phi(G) & \longrightarrow & G & \longrightarrow & G/\Phi(G) & \longrightarrow & 1
\end{array}
\]

This has exact rows and columns (the left map is onto because \( \varphi([F,F]^p) = [G,G]^p \)). The snake lemma is valid in the category of groups whenever the vertical maps have cokernels (the proof is the same). We then have an exact sequence \( 1 \rightarrow \ker \overline{\varphi}|_{\Phi(F)} \rightarrow \ker \varphi \rightarrow \ker \overline{\varphi} \rightarrow 1 \), so \( \ker \varphi \) surjects onto \( \ker \overline{\varphi} \) which is the same as saying that \( R \) generates \( M/\Phi(F) \).

Now let \( r_1 \Phi(F), ..., r_k \Phi(F) \) be a basis for \( M/\Phi(F) \) where \( r_1, ..., r_k \in R \). By linear algebra \( k = d(F) - d(G) \). Extend \( r_1 \Phi(F), ..., r_k \Phi(F) \) to a basis \( r_1 \Phi(F), ..., r_k \Phi(F), y_1 \Phi(F), ..., y_m \Phi(F) \) of \( F/\Phi(F) \). Then \( r_1, ..., r_k, y_1, ..., y_m \) generate \( F \) by the property of \( \Phi(F) \), and since a finitely generated free group is Hopfian this must be a free basis for \( F \). Hence we have a free pro-\( p \) group \( F' = \hat{F} < y_1, ..., y_m >= F/\langle r_1, ..., r_k \rangle \), so that \( d(F') = d(F) - k = d(G) \), and \( F' \) surjects on \( G \). Take \( S \) as the image of \( R \setminus \{r_1, ..., r_k\} \) in \( F' \); then \( |S| \leq |R| - k \), and we can add trivial relations to make this an equality. Then \( F' \rightarrow G \) is a minimal presentation and by construction \( |S| - d(F') = |R| - k - (d(F) - k) = |R| - d(F) \).

\( \square \)

Lemma 2. Let \( \Gamma \) be a discrete group, \( <X;R> \) a presentation of \( \Gamma \) with \(|X|\) finite. Then \( <X;R> \) is also a presentation for its pro-\( p \) completion \( \hat{\Gamma} \).

Proof. We check this has the universal property of the completion. To see this, we write \( F = F(X) \) as the discrete free group on the elements of \( X \), and \( \hat{F} \) for its pro-\( p \) completion. We can then write \( \Gamma = \hat{F} / \langle <R> \rangle \). Suppose \( G \) is a pro-\( p \) group and we have a map \( F/\langle R \rangle \rightarrow G \). One has a commutative diagram consisting of \( F \rightarrow F/\langle R \rangle \rightarrow G \), thus giving a unique map from \( \hat{F} \) and hence a unique map from \( \hat{F}/\langle R \rangle \), which is what we wanted to prove.

In a diagram this looks like this:

3
Theorem. Let $\Gamma$ be a discrete group with a finite presentation $\langle X; R \rangle$. Assume $\hat{\Gamma}_p$ is $p$-adic analytic but not isomorphic to $\mathbb{Z}_p$. Then

$$|R| - (|X| - d(\hat{\Gamma}_p)) \geq d(\hat{\Gamma}_p)^2/4$$

Proof. By Lemma 2, the given presentation is also a presentation for $\hat{\Gamma}_p$. By Lemma 1, we can replace the given presentation by a minimal presentation of $\hat{\Gamma}_p$ with $s = |R| - (|X| - d(\hat{\Gamma}_p))$ relations.

The extension of the Golod-Shafarevich theorem now gives

$$|R| - (|X| - d(\hat{\Gamma}_p)) = s \geq \frac{d(\hat{\Gamma}_p)^2}{4}$$

4 Minimal presentations of finitely generated nilpotent groups

Proposition 1. For any finitely generated group $\Gamma$ there is a prime $p$ such that $d_{ab}(\Gamma) = d(\hat{\Gamma}_p)$.

Proof. First we claim that $d(\hat{\Gamma}_p) = d(\Gamma/[\Gamma, \Gamma]\Gamma^p)$.

Let $H$ be the Frattini subgroup of $\hat{\Gamma}_p$, i.e. $H = [\hat{\Gamma}_p, \hat{\Gamma}_p]^{\mathbb{F}_p}$. We then have $\hat{\Gamma}_p/H \simeq (\Gamma/[\Gamma, \Gamma]\Gamma^p)_{\mathbb{F}_p}$, because completion commutes with the quotient by a normal subgroup. Notice now that $\Gamma/[\Gamma, \Gamma]\Gamma^p$ is just a finite abelian $p$-group, so its completion is isomorphic to itself. It follows that $d(\hat{\Gamma}_p) = d(\Gamma_p/H) = d((\Gamma/[\Gamma, \Gamma]\Gamma^p)_{\mathbb{F}_p}) = d(\Gamma/[\Gamma, \Gamma]\Gamma^p)$.

Now write $\Gamma/[\Gamma, \Gamma] \simeq \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_r\mathbb{Z} \oplus \mathbb{Z}^k$, where $d_1 | \ldots | d_r$ so $d_{ab}(\Gamma) = r + k$. Let $p$ be such that $p | d_1$ (or $p = 2$ if $\Gamma/[\Gamma, \Gamma]$ is torsion free), then clearly $\Gamma/[\Gamma, \Gamma]\Gamma^p$ is a vector space of dimension $r+k$ over $\mathbb{F}_p$, hence also $d(\Gamma/[\Gamma, \Gamma]\Gamma^p) = r + k$.

So in conclusion, we have $d(\hat{\Gamma}_p) = d(\Gamma/[\Gamma, \Gamma]\Gamma^p) = r + k = d_{ab}(\Gamma)$. 

□
We want to say something strong on nilpotent groups. For this we will do two things: to show that in that case every \( \hat{\Gamma}_p \) is \( p \)-adic analytic and to relate \( d_{ab}(\Gamma) \) with \( d(\Gamma) \).

**Lemma (without proof).** If \( 1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1 \) is an exact sequence with \( \hat{\Gamma}_p, \hat{\Gamma}'_p, \hat{\Gamma}''_p \) \( p \)-adic analytic, then so is \( \hat{\Gamma}_p \).

**Proposition 2.** Let \( \Gamma \) be a finitely generated solvable noetherian group (every subgroup is finitely generated). Then \( \hat{\Gamma}_p \) is \( p \)-adic analytic for every \( p \).

**Proof.** Since \( \Gamma \) is solvable, there is a series of subgroups \( 1 \triangleleft N_1 \triangleleft \ldots \triangleleft N_k \triangleleft \Gamma \), with each \( N_i / N_{i+1} \) abelian. Since \( \Gamma \) is noetherian, each \( N_i / N_{i+1} \) is finitely generated, so we can refine this enough so that each \( N_i / N_{i+1} \) is cyclic. For a cyclic group it is clear that its pro-\( p \) completion is always \( p \)-adic analytic, so by the previous lemma we get that if for \( N_k \) the pro-\( p \) completion is \( p \)-adic analytic then so is \( \hat{\Gamma}_p \). Now use induction on the length of the composition series.

**Lemma 3.** Let \( \Gamma \) be a finitely generated nilpotent group, then \( d(\Gamma) = d_{ab}(\Gamma) \).

**Proof.** Since \( d(\Gamma) = d(\Gamma/\Phi(\Gamma)) \), \( d(\Gamma_{ab}) = d(\Gamma_{ab}/\Phi(\Gamma_{ab})) \), it is enough to show that \( \Gamma/\Phi(\Gamma) \cong \Gamma_{ab}/\Phi(\Gamma_{ab}) \). Recall that an epimorphism \( G \to H \) always factors to a map \( G/\Phi(G) \to H/\Phi(H) \). It is known that for a nilpotent group, \( [\Gamma, \Gamma] \subset \Phi(\Gamma) \). Therefore there is an epimorphism \( \Gamma_{ab} \to \Gamma/\Phi(\Gamma) \) which factors to a map \( \Gamma_{ab}/\Phi(\Gamma_{ab}) \to \Gamma/\Phi(\Gamma) \). On the other hand, the epimorphism \( \Gamma \to \Gamma_{ab} \) factors to a map \( \Gamma/\Phi(\Gamma) \to \Gamma_{ab}/\Phi(\Gamma_{ab}) \), and these two maps are clearly inverses.

**Theorem.** If \( \Gamma \) is a finitely generated nilpotent group different from \( \mathbb{Z} \), and \( \langle X; R \rangle \) is a minimal presentation for \( \Gamma \), then \( |R| \geq \frac{|X|^2}{4} \).

**Proof.** It is known that a finitely generated nilpotent group is solvable and noetherian. Hence by the previous proposition each \( \hat{\Gamma}_p \) is \( p \)-adic analytic. Moreover, there is some \( p \) such that \( d(\hat{\Gamma}_p) = d_{ab}(\Gamma) \), which equals \( d(\Gamma) \) by the previous lemma. Moreover, if \( \hat{\Gamma}_p = \mathbb{Z}_p \), then \( d(\hat{\Gamma}_p) = d(\Gamma) = 1 \), which is possible only if \( \Gamma \) is cyclic. In that case we can’t have \( \hat{\Gamma} = \mathbb{Z} \), so \( \Gamma \) is finite and cyclic in which case the inequality clearly holds. Otherwise, by the previous theorem we have \( |R| - (|X| - d(\Gamma)) \geq d(\Gamma)^2/4 \). But \( d(\Gamma) = |X| \) by definition, so \( |R| \geq \frac{|X|^2}{4} \).