GEOMETRIC DESCRIPTION OF PERIOD RINGS

GAL PORAT

Abstract. These are notes which aim to give a short summary of the geometric description of some of the various period rings appearing in $p$-adic Hodge theory.

For a more thorough discussion see Scholze and Weinstein’s Berkeley notes. Our notation for the rings follows Berger’s normalization.

1. THE PERIOD RINGS

We shall work here with $\mathbb{Q}_p$. In general, one can work with a finite extension of $\mathbb{Q}_p$.

1.1. The ring $\mathbb{C}^\flat_p$. Let $\mathcal{O}_{\mathbb{C}^\flat_p} = \lim_{x \leftarrow x^p} \mathcal{O}_{\mathbb{C}^p}/\mathbb{Z}_p$ and $\mathbb{C}^\flat_p$ be its ring of fractions. It carries an action of $G_{\mathbb{Q}_p} = \text{Gal}(\mathbb{Q}_p/\mathbb{Z}_p)$. There’s a multiplicative map $\sharp: \mathbb{C}^\flat_p \rightarrow \mathbb{C}^p$ which sends an element $(\overline{x}_0, \overline{x}_1, \ldots)$ to $\lim_{n \rightarrow \infty} x_n^p$; in other words, it sends an element in $\mathcal{O}_{\mathbb{C}^\flat_p}$ to its zeroth $\mathcal{O}_{\mathbb{C}^p}$-coordinate. This allows us to define a valuation by $|x|_{\mathbb{C}^\flat_p} = |x^\sharp|_{\mathbb{C}^p}$.

If $x \in \mathbb{C}_p$, we let $x^\flat \in \mathbb{C}^\flat_p$ denote some element so that $(x^\flat)^\sharp$. It is very much not unique (though the ambiguity can be described in a precise way). For example, $p^\flat = (1, \zeta_p, \zeta_p^2, \ldots)$.

Other notations: the ring $\mathcal{O}_{\mathbb{C}^\flat_p}$ (resp. $\mathbb{C}^\flat_p$) is sometimes denoted $\tilde{E}^+$ (resp. $\tilde{E}$) by Berger and Colmez.

1.2. The ring $A_{\text{inf}}$. We set $A_{\text{inf}} = W\left(\mathcal{O}_{\mathbb{C}^\flat_p}\right)$. This ring has a lot of structure. On the one hand, we endow it with the $(p, [p^\flat])$-topology, which is sometimes also called the weak topology. Every element in $A_{\text{inf}}$ has a unique Teichmüller expansion of the form $\sum_{n \geq 0} [x_n] p^n$ with $x_n \in \mathcal{O}_{\mathbb{C}^\flat_p}$, though working out the ring operations in terms of these ring expansions has to do with Witt polynomials and can be a bit messy. On the other hand, it has Frobenius map $\varphi: A_{\text{inf}} \rightarrow A_{\text{inf}}$ given by mapping $\varphi([x_n]) = [x_n^p]$, as well as a $G_{\mathbb{Q}_p}$ action induced from its action on $\mathbb{C}^\flat_p$. There is also Fontaine’s map $\theta: A_{\text{inf}} \rightarrow \mathbb{C}^p$ given by

$$\sum_{n \geq 0} [x_n] p^n \mapsto \sum_{n \geq 0} x_n^\sharp p^n.$$
It can be shown that \( \ker \theta = (p - [p^s]) = \left( \frac{[1^s] - 1}{(1^{1/p})^s - 1} \right). \) Sometimes one writes \( \xi = p - [p^s] \) and \( \omega = \frac{[1^s] - 1}{(1^{1/p})^s - 1}. \) The ker \( \theta \)-topology is stronger than the \((p, [p^s])\)-topology, because ker \( \theta \) is strictly contained in \((p, [p^s])\).

**Other notations:** the ring \( A_{\text{inf}} \) is sometimes denoted \( \tilde{A}^+ \) by Berger and Colmez.

### 1.3. The rings \( B_{\text{dr}}^+ \) and \( B_{\text{dR}}. \)

The ring \( B_{\text{dr}}^+ \) is defined to be the ker \( \theta \)-completion of \( A_{\text{inf}}, \) and \( B_{\text{dR}} = \text{Frac}(B_{\text{dr}}^+). \) One can show that \( t = \log [1^s] \) exists and is a uniformizer of \( B_{\text{dr}}^+ \), making it into a DVR and endowing it with a filtration \( \text{Fil}^i B_{\text{dr}} = t^i B_{\text{dr}}^+. \)

### 1.4. The rings \( \tilde{A}' \) and \( \tilde{B}' \).

(See section 2 of Berger’s thesis). Let \( \varpi \) be a pseudouniformizer of \( O_{F_p} \), with valuation \( |\varpi|_p = p^{-\frac{p-1}{p-1}}. \) For example, you could take \( \varpi = \left( \frac{p^p}{p^{p-1}} \right)^b = (p^{p-1/p}, p^{p-1/p^2}, \ldots) \) or \( \varpi = (\zeta_p - 1)^b, \) so that \( \varpi = (0, \zeta_p - 1, \zeta_p^2 - 1, \ldots). \) (The latter element is sometimes dubbed \( \varepsilon - 1 \) by Berger or Colmez). Given \( r \leq s \in \mathbb{Z}_{\geq 0}/1/p \) and \( I = [r, s], \) one sets

\[
\tilde{A}^{[r,s]} = A_{\text{inf}} \langle \frac{p}{\varpi^r}, \frac{[\varpi]^s}{p} \rangle.
\]

Here, the notation \( \langle \cdot \rangle \) means completion with respect to the \((p, [\varpi]) = (p, [p^s])\)-topology. This is the same as completion with respect to the \( p \)-adic topology, because \( p \) divides a power of \([\varpi]\) in this ring.

We also define

\[
\tilde{A}^{[r,\infty]} = A_{\text{inf}} \langle \frac{p}{\varpi^r} \rangle, \tilde{A}^{[\infty,\infty]} = A_{\text{inf}} \langle \frac{1}{\varpi} \rangle,
\]

where the completion is taken with respect to the \( p \)-adic topology.

In all of these cases we may define \( \tilde{B}' = \tilde{A}'[1/p]. \)

When \( [r_2, s_2] \subset [r_1, s_1] \) there are injective maps \( \tilde{A}^{[r_1,s_1]} \to \tilde{A}^{[r_2,s_2]} \) and \( \tilde{B}^{[r_1,s_1]} \to \tilde{B}^{[r_2,s_2]} \).

Finally, for a general interval we can set \( \tilde{A}' = \cap_{[r,s] \subset I} \tilde{A}^{[r,s]} \) and \( \tilde{B}' = \tilde{A}'[1/p]. \)

These rings can also be defined by a valuation. Any element of \( A_{\text{inf}}[1/p [\varpi]] \) can be written uniquely in the form \( \sum_{n \gg -\infty} x_n p^n. \) If \( I \) is an interval, set \( V_I \) for the valuation given by

\[
V_I \left( \sum_{n \gg -\infty} x_n p^n \right) = \inf_{r \in I} \inf_{n \in \mathbb{Z}} \left( n + \frac{p - 1}{p^{r}} \text{val}_p (x_n) \right).
\]

Then one can define \( \tilde{A}' \) as the ring of integers of \( \tilde{B}', \) where we define

\[
\tilde{B}' = \left\{ \begin{array}{ll}
\text{The completion of } A_{\text{inf}}[1/p [\varpi]] \text{ with respect to } V_I & 0 \notin I \\
\text{The completion of } A_{\text{inf}}[1/p] \text{ with respect to } V_I & 0 \in I.
\end{array} \right.
\]

The Frobenius map \( \varphi : A_{\text{inf}} \to A_{\text{inf}} \) induces maps \( \tilde{A}' \to \tilde{A}^p \) and \( \tilde{B}' \to \tilde{B}^p. \)
1.5. **The rings** $A^+_{\text{max}}, B^+_{\text{max}}$ and $B_{\text{max}}$. We set $r_n = p^{n-1}(p-1)$ so that $r_0 = \frac{p-1}{p}$. Then we define

$$B^+_{\text{max}} = \widehat{\mathbb{B}}[0, r_0] = \widehat{\mathbb{B}}[0, \frac{p-1}{p}] = A_{\text{inf}} \left[ \frac{[\pi^{p-1}_p]}{p} \right] \left[ 1/p \right] = A_{\text{inf}} \left( \frac{[p^\flat]}{p} \right) \left[ 1/p \right]$$

and similarly $A^+_{\text{max}} = \widehat{\mathbb{A}}[0, r_0]$. The ring $B^+_{\text{max}}$ admits a map to $B_{\text{dR}}^+$. Indeed, we have the identity

$$\frac{[p^\flat]}{p} = 1 + \left( \frac{[p^\flat] - p}{p} \right) \equiv 1 \mod \ker \theta,$$

so sequences $\sum_n a_n \left( \frac{[p^\flat]}{p} \right)^n$ with $a_n \rightarrow 0$ in the $(p, [p^\flat])$-topology also have $a_n \left( \frac{[p^\flat]}{p} \right)^n \rightarrow 0$ in the ker $\theta$-topology.

The rings $A^+_{\text{max}}, B^+_{\text{max}}$ and $B_{\text{max}}$ have an action of $\varphi$ via the composition $\widehat{\mathbb{B}}[0, r_0] \xrightarrow{\varphi} \widehat{\mathbb{B}}[0, p r_0] \hookrightarrow \widehat{\mathbb{B}}[0, r_0]$.

The element $t = \log [1^\flat]$ makes sense in $B^+_{\text{max}}$. Indeed, we have $[1^\flat] - 1 \in \ker \theta = (p - [p^\flat])$, so that $\frac{([1^\flat]-1)_n}{n} = (\text{integral}) \frac{p^n \left( \frac{[p^\flat]}{p} \right)}{n} \rightarrow 0$ in $A_{\text{inf}} \left( \frac{[p^\flat]}{p} \right)$. It follows that

$$t = \log [1^\flat] = \sum_{n \geq 1} (-1)^{n-1} \frac{([1^\flat]-1)_n}{n}$$

converges in $A_{\text{inf}} \left( \frac{[p^\flat]}{p} \right) \left[ 1/p \right]$.

1.6. **The rings** $A^+_{\text{cris}}, B^+_{\text{cris}}$ and $B_{\text{cris}}$. The ring $A^+_{\text{cris}}$ is defined to be the $p$-adic completion of $A_{\text{inf}} \left[ \frac{(p-[p^\flat])_n}{n!} \right]_{n \geq 1}$. We define $B^+_{\text{cris}} = A^+_{\text{cris}}[1/p]$. Since $[1^\flat] - 1 \in \ker \theta = (p - [p^\flat])$, we have

$$\frac{([1^\flat]-1)_n}{n} = (n-1)! \text{(integral)} \frac{(p-[p^\flat])_n}{n!}$$

so that $t = \log [1^\flat]$ belongs to $B^+_{\text{cris}}$. We set $B_{\text{cris}} = B^+_{\text{cris}}[1/t]$.

The ring $B^+_{\text{cris}}$ is very close to being equal to $\widehat{\mathbb{B}}[0, \frac{(p-[p^\flat])^2}{p}] = A_{\text{inf}} \left( \frac{[p^\flat]}{p} \right) \left[ 1/p \right]$. On the one hand, we claim there is a containment $B^+_{\text{cris}} \subset \widehat{\mathbb{B}}[0, \frac{(p-[p^\flat])^2}{p}]$. Indeed, given $n$ we will show that $\frac{(p-[p^\flat])_n}{n!} \in \widehat{\mathbb{B}}[0, \frac{(p-[p^\flat])^2}{p}]$. Write $n = k(p-1) + r$ with $0 \leq r \leq p-2$, and remember that $v_p(n!) = \frac{n - s_p(n)}{p-1} = k + \frac{r - s_p(n)}{p-1}$, where $s_p(n)$ is the sum of digits of $n$ in base $p$. Then we have

$$\frac{(p-[p^\flat])_n}{n!} = (\text{unit}) \left( \frac{(p-[p^\flat])^{p-1}}{p} \right)^k p^{\frac{n-s_p(n)-r}{p-1}} (p-[p^\flat])^r.$$
Now, we see that \( \left( \frac{(p-[p])}{p} \right)^{p-1} \) belongs to \( \bar{B}^{[0,(p-1)^2/p]} \), as well as \( (p-[p])^r \). On the other hand \( \frac{s_p(n)-r}{p-1} \geq \frac{p-2}{p-1} > -1 \), but it is also an integer, so \( \frac{s_p(n)-r}{p-1} \geq 0 \). It follows that \( p^{\frac{s_p(n)-r}{p-1}} \) is a positive power of \( p \), so we see that the entire product, which is \( \frac{(p-[p])^n}{n!} \), lies in \( \bar{B}^{[0,(p-1)^2/p]} \).

It is \( p \)-adically complete, so \( B_{\text{cris}}^+ \subset \bar{B}^{[0,(p-1)^2/p]} \).

In the other direction, it is almost true that \( \bar{B}^{[0,(p-1)^2/p]} = A_{\text{inf}} \left( \frac{[p]^{(p-1)+pr}}{p} \right) [1/p] \subset B_{\text{cris}}^+ \).

However there are some rationality problems, i.e. \( B_{\text{cris}}^+ \) only contains some power of the variable \( \frac{[p]^{(p-1)+pr}}{p} \). Indeed, give such an \( r \), take \( n \) divisible by \( p-1 \) and such that \( s_p((p-1)n) \leq prn \). This is possible because \( s_p \) grows logarithmically. We then have

\[
\left( \frac{p-[p]}{p} \right)^{(p-1)n} = p^{s_p(n)} \left( \frac{(p-[p])^{p-1}}{p} \right)^n \equiv \mod A_{\text{inf}} p^{s_p(n)} \left( \frac{[p]^{p-1}}{p} \right)^n.
\]

This shows that \( p^{s_p(n)} \left( \frac{[p]^{p-1}}{p} \right)^n \in B_{\text{cris}}^+ \). This element divides \( p^{prn} \left( \frac{[p]^{p-1}}{p} \right)^n = \left( \frac{[p]^{(p-1)+pr}}{p} \right)^n \), so we see that \( \left( \frac{[p]^{(p-1)+pr}}{p} \right)^n \in B_{\text{cris}}^+ \) as claimed. Making this argument a little more precise will also show that \( \left( \frac{[p]^{(p-1)+pr}}{p} \right)^n \in B_{\text{cris}}^+ \) for all \( n >> 0 \).

We summarize the above discussion in the following proposition.

**Proposition 1.1.** We have \( B_{\text{cris}}^+ \subset \bar{B}^{[0,(p-1)^2/p]} \). For any \( r > 0 \) there exists a finite map \( \bar{B}^{[0,(p-1)^2/p]} \to B_r = \text{such that } B_r \subset B_{\text{cris}}^+ \).

We notice a few more things. The first is that \( \bar{B}^{[0,p-1]} \subset B_{\text{cris}}^+ \) (no need for a finite extension). The reason is that the coordinate of \( \bar{B}^{[0,p-1]} \) is given by \( \frac{[p]^p}{p} \), which is equivalent \( \mod A_{\text{inf}} \) to a unit times \( \frac{[p]^p}{p^l} \). It then follows from some algebra of divided powers that all powers of \( \frac{[p]^p}{p} \) lie in \( B_{\text{cris}}^+ \). The ring \( B_{\text{cris}}^+ \) is endowed with a Frobenius map, because of an identity having to do with divided powers. Moreover, we have

\[
\varphi(B_{\text{max}}^+) = \bar{B}^{[0,p-1]} \subset B_{\text{cris}}^+ \subset \bar{B}^{[0,(p-1)^2/p]} \subset \bar{B}^{[0,(p-1)^2/p]} = B_{\text{max}}^+,
\]

so another way we can think of \( \varphi \) is by being the map induced from \( B_{\text{cris}}^+ \to B_{\text{max}}^+ \to \varphi(B_{\text{max}}^+) \to B_{\text{cris}}^+ \).

1.7. **The rings** \( \bar{B}_{\text{rig}}^+, \bar{B}_{\text{rig}}^{t,r} \text{ and } \bar{B}_{\text{rig}}^{t,l} \). We set \( \bar{B}_{\text{rig}}^+ = \bar{B}^{[0,\infty]} \), \( \bar{B}_{\text{rig}}^{t,r} = \bar{B}^{[r,\infty]} \) and \( \bar{B}_{\text{rig}}^{t,l} = \bigcup_{r \geq 0} \bar{B}_{\text{rig}}^{t,r} \).

The Frobenius map induces \( \varphi : \bar{B}_{\text{rig}}^{t,r} \to \bar{B}_{\text{rig}}^{t,r} \) and hence \( \varphi : \bar{B}_{\text{rig}}^+ \to \bar{B}_{\text{rig}}^+ \) and \( \varphi : \bar{B}_{\text{rig}}^{t,l} \to \bar{B}_{\text{rig}}^{t,l} \).
GEOMETRIC DESCRIPTION OF PERIOD RINGS

It is useful to note that \( \overline{B}_{\text{rig}}^+ = \cap_{n \geq 1} \varphi^{-n}(B_{\text{max}}^+) = \cap_{n \geq 1} \varphi^{-n}(B_{\text{cris}}^+) \), which makes \( \varphi \) into an automorphism of \( \overline{B}_{\text{rig}}^+ \).

2. THE GEOMETRIC SPACES

The spaces we shall work with here are adic or pre-adic spaces. The specific formalism of adic spaces is not too important for us, but if \( (R, R^+) \) is a Huber pair, \( \text{Spa}(R, R^+) \) is a space whose points correspond to valuations and whose functions are basically \( R \). The space \( \text{Spa}(R, R^+) \) is not always a locally ringed space (the structure presheaf is not always a sheaf), and that’s the distinction between being a pre-adic and an adic space. If \( x \) is a point, we denote the valuation by \( f \mapsto |f(x)| \). There is a well defined operation of evaluating at a point: if \( x \) is a point, then the kernel of its valuation is a prime ideal of \( R \), and so we think of \( \text{Frac}(R/\ker | \cdot |) \) as being the residue field at \( x \). Finally, we note that these valuations may be valued in strange groups, but the operation of “maximal generization” always returns a point whose valuation is valued in \( \mathbb{R}_{\geq 0} \) (see section 4.2 of the Berkeley notes). Recall also an analytic point is a valuation whose kernel is nonopen.

Let \( A_{\text{inf}} \) be as in section 1. The space \( \text{Spa}(A_{\text{inf}}) = \text{Spa}(A_{\text{inf}}, A_{\text{inf}}) \) is a pre-adic space. It is probably also an adic space but it’s not clear if that is known at the moment (see footnote in section 12 of Berkeley notes). In \( \text{Spa}(A_{\text{inf}}) \), it is useful to denote four special points by their residue fields.

1. \( x_{\mathbb{F}_p} \) is the unique non-analytic point, given by \( A_{\text{inf}} \to A_{\text{inf}}/(p, [p^\flat]) = \mathbb{F}_p \).
2. \( x_{\mathbb{C}_p} \) is given by \( A_{\text{inf}} \to A_{\text{inf}}/p = \mathcal{O}_{\mathbb{C}_p} \to \mathbb{C}_p^\flat \).
3. \( x_{\mathbb{C}_p} \) is given by \( A_{\text{inf}} \to A_{\text{inf}}/(p - [p^\flat]) = \mathcal{O}_{\mathbb{C}_p} \to \mathbb{C}_p \).
4. \( x_{\mathbb{Q}_p} \) is given by \( A_{\text{inf}} \to A_{\text{inf}}/ [p^\flat] = \mathbb{Z}_p \to \mathbb{Q}_p \).

This is basically how all points look like, at least those which correspond to close prime ideals. See Colmez’s survey on the Fargues-Fontaine curve, corollary 3.3.

We let \( \mathcal{Y} = \text{Spa}A_{\text{inf}} - \{ x_{\mathbb{F}_p} \} \), which is known to be an analytic adic space. There exists a surjective continuous map \( \kappa : \mathcal{Y} \to [0, \infty] \), given by

\[
\kappa(x) = \frac{\log |p^\flat(\overline{x})|}{\log |p(x)|}.
\]

We have \( \kappa \circ \varphi = p\kappa \). In particular,

\[
\kappa\left( x_{\mathbb{C}_p} \right) = 0, \kappa\left( x_{\mathbb{C}_p} \right) = 1, \kappa\left( \varphi^n (x_{\mathbb{C}_p}) \right) = p^n, \kappa\left( x_{\mathbb{C}_p} \right) = \infty.
\]

This can be seen in the following picture, from page 101 of the Scholze-Weinstein notes.
For an interval $I \subset [0, \infty]$, we let $Y_I$ be the interior of the preimage of $Y$ under $\kappa$.

We visualize $Y$ as being a sphere, and $\kappa = 0$ and $\kappa = \infty$ correspond to two opposing poles of this sphere. In the middle $\kappa = r$ represent circles lying in between.

With this in place, we may now give a geometric interpretation for the various rings appearing above.

2.1. **The ring $C^p$.** The ring $C^p$ is the residue ring of $x_{C^p}$ in $Y$. Thus $C^p = k\left(x_{C^p}\right)$.

2.2. **The ring $A_{\text{inf}}$.** The ring $A_{\text{inf}}$ is the coordinate ring of $\text{Spa}A_{\text{inf}}$.

2.3. **The rings $B^+_{dR}$ and $B_{dR}$.** The ring $B^+_{dR}$ is the completion of the local ring of $x_{C^p}$. Thus $B^+_{dR} = \hat{O}_{Y,x_{C^p}}$. We may think of $t = \log\left[1^p\right]$ as giving a choice of a local coordinate. Thus if we are given a function which is defined in $x_{C^p}$, it has an image in $B^+_{dR}$, and this image is its Taylor expansion at $x_{C^p}$ in terms of $t$. Evaluating this function gives an element of $C_p$, which is the same as taking the image through the homomorphism $B^+_{dR} \to B^+_{dR}/t$.

2.4. **The ring $\tilde{B}'$.** Let $\rho(r) := \frac{p^r}{p^r}$ and $\rho(\infty) = 0$. This operation reverses directions between $r$ and $s$ and renormalizes. If $I = [r, s]$, let $\rho(I) = [\rho(s), \rho(r)]$. The ring $\tilde{B}'$ is none other than $H^0\left(Y_{p^{-1}I}, O_Y\right)$, the rings of functions converging on a closed rational set. In fact if $I$ is closed (maybe also need $0 \notin I$), this set is an affinoid (proved by Kedlaya and Liu), so can be thought of as a coordinate ring.

Since we think of $Y$ as a sphere, if $I \subset (0, \infty)$ it's useful to think of this as being the ring of functions converging on some annuli.
2.5. **The ring** $B_{\text{max}}^+$. We have $\rho \left( \left[ 0, \frac{p-1}{p} \right] \right) = [1, \infty]$, so $B_{\text{max}}^+$ is the same as $H^0 \left( \mathcal{Y}_{[1,\infty]}, \mathcal{O}_Y \right)$. Thus it is the ring of functions on a closed disc.

2.6. **The ring** $B_{\text{cris}}^+$. It follows from the discussion before that $B_{\text{cris}}^+$ is something like the ring of functions on a space, which on the one hand covers the disc $\mathcal{Y}_{[\frac{1}{p-1}, \infty]}$, and on the other hand for $r > 0$ admits cover by a finite covering of the slightly larger disc $\mathcal{Y}_{[r, \infty]}$ for $r < \frac{1}{p-1}$.

2.7. **The rings** $\tilde{B}_{\text{rig}}^+, \tilde{B}_{\text{rig}}^{+r}$ and $\tilde{B}_{\text{rig}}^t$. The ring $\tilde{B}_{\text{rig}}^+$ is the ring of functions $H^0 \left( \mathcal{Y}_{(0,\infty]}, \mathcal{O}_Y \right)$, the ring $\tilde{B}_{\text{rig}}^{+r}$ is the ring of functions $H^0 \left( \mathcal{Y}_{(0,\frac{1}{p-1}]}, \mathcal{O}_Y \right)$ (on a punctured annulus), and $\tilde{B}_{\text{rig}}^t$ is the local ring on at the puncture of 0, in other words it’s the local ring $\mathcal{O}_{\mathcal{Y},x_{C_p}}$ except that we also allows arbitrary poles at $x_{C_p}$. 