GEOMETRIC DESCRIPTION OF PERIOD RINGS

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Abstract. These are notes which aim to give a short summary of the geometric description of some of the various period rings appearing in p-adic Hodge theory.

For a more thorough discussion see Scholze and Weinstein's Berkeley notes. Our notation for the rings follows Berger's normalization.

1. The period rings

We shall work here with $\mathbb{Q}_p$. In general, one can work with a finite extension of $\mathbb{Q}_p$.

1.1. The ring $\mathbb{C}_p^\flat$.

Let $\mathcal{O}_{\mathbb{C}_p^\flat} = \lim_{x^p \leftarrow x} \mathcal{O}_{\mathbb{C}_p}/p \cong \lim_{x^p \leftarrow x} \mathcal{O}_{\mathbb{C}_p}$ and $\mathbb{C}_p^\flat$ be its ring of fractions. It carries an action of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. There's a multiplicative map $\sharp : \mathbb{C}_p^\flat \to \mathbb{C}_p$ which sends an element $(\overline{x}_0, \overline{x}_1, ...)$ to $\lim_{n\to\infty} x_n^{p^n}$; in other words, it sends an element in $\mathcal{O}_{\mathbb{C}_p^\flat}$ to its zeroth $\mathcal{O}_{\mathbb{C}_p}$-coordinate. This allows us to define a valuation by $|x|_{\mathbb{C}_p^\flat} = |x^\sharp|_{\mathbb{C}_p}$.

If $x \in \mathbb{C}_p$, we let $x^\flat \in \mathbb{C}_p^\flat$ denote some element so that $\left(x^\flat\right)^\sharp$. It is very much not unique (though the ambiguity can be described in a precise way). For example, $p^\flat = (p, p^{1/p}, ...)$ and $1^\flat = (1, \zeta_p, \zeta_{p^2}, ...)$.

Other notations: the ring $\mathcal{O}_{\mathbb{C}_p^\flat}$ (resp. $\mathbb{C}_p^\flat$) is sometimes denoted $\widetilde{\mathrm{E}}^+$ (resp. $\widetilde{\mathrm{E}}$) by Berger and Colmez.

1.2. The ring $\mathrm{A}_{\mathrm{inf}}$.

We set $\mathrm{A}_{\mathrm{inf}} = \mathrm{W}\left(\mathcal{O}_{\mathbb{C}_p^\flat}\right)$. This ring has a lot of structure. On the one hand, we endow it with the $\left(p, \left[p^\flat\right]\right)$-topology, which is sometimes also called the weak topology. Every element in $\mathrm{A}_{\mathrm{inf}}$ has a unique Teichmuller expansion of the form $\sum_{n\geq 0}[x_n]p^n$ with $x_n \in \mathcal{O}_{\mathbb{C}_p^\flat}$, though working out the ring operations in terms of these ring expansions has to do with Witt polynomials and can be a bit messy. On the other hand, it has Frobenius map $\varphi : \mathrm{A}_{\mathrm{inf}} \to \mathrm{A}_{\mathrm{inf}}$ given by mapping $\varphi([x_n]) = [x_n^p]$, as well as a $G_{\mathbb{Q}_p}$ action induced from its action on $\mathbb{C}_p^\flat$. There is also Fontaine's map $\theta : \mathrm{A}_{\mathrm{inf}} \to \mathbb{C}_p$ given by

$$\sum_{n\geq 0}[x_n]p^n \mapsto \sum_{n\geq 0} x_n^\sharp p^n.$$

It can be shown that $\ker\theta = \left(p - \left[p^\flat\right]\right) = \left(\frac{\left[1^\flat\right]-1}{\left[\left(1^\flat\right)^{1/p}\right]-1}\right)$. Sometimes one writes $\xi = p - \left[p^\flat\right]$ and $\omega = \frac{\left[1^\flat\right]-1}{\left[\left(1^\flat\right)^{1/p}\right]-1}$. The $\ker\theta$-topology is stronger than the $\left(p, \left[p^\flat\right]\right)$-topology, because $\ker\theta$ is strictly contained in $\left(p, \left[p^\flat\right]\right)$.

Other notations: the ring $\mathrm{A}_{\mathrm{inf}}$ is sometimes denoted $\widetilde{\mathrm{A}}^+$ by Berger and Colmez.

1.3. **The rings $\mathrm{B}_{\mathrm{dR}}^+$ and B_{dR}.** The ring $\mathrm{B}_{\mathrm{dR}}^+$ is defined to be the $\ker\theta$-completion of $\mathrm{A}_{\mathrm{inf}}$, and $\mathrm{B}_{\mathrm{dR}} = \mathrm{Frac}(\mathrm{B}_{\mathrm{dR}}^+)$. One can show that $t = \log\left[1^\flat\right]$ exists and is a uniformizer of $\mathrm{B}_{\mathrm{dR}}^+$, making it into a DVR and endowing it with a filtration $\mathrm{Fil}^i\mathrm{B}_{\mathrm{dR}} = t^i\mathrm{B}_{\mathrm{dR}}^+$.

1.4. **The rings $\widetilde{\mathrm{A}}^I$ and $\widetilde{\mathrm{B}}^I$.** (See section 2 of Berger's thesis). Let ϖ be a pseudouniformizer of $\mathcal{O}_{\mathbb{C}_p^\flat}$ with valuation $|\varpi|_\flat = p^{-\frac{p}{p-1}}$. For example, you could take $\varpi = \left(p^{\frac{p}{p-1}}\right)^\flat = (p^{p-1/p}, p^{p-1/p^2}, ...)$ or $\varpi^{\frac{1}{p}} = (\zeta_p - 1)^\flat$, so that $\varpi = \left(0, \overline{\zeta_p - 1}, \overline{\zeta_{p^2} - 1}, ...\right)$. (The latter element is sometimes dubbed $\varepsilon - 1$ by Berger or Colmez). Given $r \leq s \in \mathbb{Z}_{\geq 0}[1/p]$ and $I = [r, s]$, one sets

$$\widetilde{\mathrm{A}}^{[r,s]} = \mathrm{A}_{\mathrm{inf}}\left\langle \frac{p}{[\varpi^r]}, \frac{[\varpi^s]}{p} \right\rangle.$$

Here, the notation $\langle\cdot\rangle$ means completion with respect to the $(p, [\varpi]) = \left(p, \left[p^\flat\right]\right)$-topology. This is the same as completion with respect to the p-adic topology, because p divides a power of $[\varpi]$ in this ring.

We also define

$$\widetilde{\mathrm{A}}^{[r,\infty]} = \mathrm{A}_{\mathrm{inf}}\left\langle \frac{p}{[\varpi^r]} \right\rangle, \widetilde{\mathrm{A}}^{[\infty,\infty]} = \mathrm{A}_{\mathrm{inf}}\left\langle \frac{1}{[\varpi]} \right\rangle,$$

where the completion is taken with respect to the p-adic topology.

In all of these cases we may define $\widetilde{\mathrm{B}}^I = \widetilde{\mathrm{A}}^I[1/p]$.

When $[r_2, s_2] \subset [r_1, s_1]$ there are injective maps $\widetilde{\mathrm{A}}^{[r_1,s_1]} \to \widetilde{\mathrm{A}}^{[r_2,s_2]}$ and $\widetilde{\mathrm{B}}^{[r_1,s_1]} \to \widetilde{\mathrm{B}}^{[r_2,s_2]}$.

Finally, for a general interval we can set $\widetilde{\mathrm{A}}^I = \cap_{[r,s]\subset I}\widetilde{\mathrm{A}}^{[r,s]}$ and $\widetilde{\mathrm{B}}^I = \widetilde{\mathrm{A}}^I[1/p]$.

These rings can also be defined by a valuation. Any element of $\mathrm{A}_{\mathrm{inf}}[1/p\,[\varpi]]$ can be written uniquely in the form $\sum_{n>>-\infty}[x_n]p^n$. If I is an interval, set V_I for the valuation given by

$$V_I\left(\sum_{n>>-\infty}[x_n]p^n\right) = \inf_{r\in I}\inf_{n\in\mathbb{Z}}\left(n + \frac{p-1}{pr}\mathrm{val}_\flat\left(x_n\right)\right).$$

Then one can define $\widetilde{\mathrm{A}}^I$ as the ring of integers of $\widetilde{\mathrm{B}}^I$, where we define

$$\widetilde{\mathrm{B}}^I = \begin{cases} \text{The completion of } \mathrm{A}_{\mathrm{inf}}[1/p\,[\varpi]] \text{ with respect to } V_I & 0 \notin I \\ \text{The completion of } \mathrm{A}_{\mathrm{inf}}[1/p] \text{ with respect to } V_I & 0 \in I \end{cases}.$$

The Frobenius map $\varphi : \mathrm{A}_{\mathrm{inf}} \to \mathrm{A}_{\mathrm{inf}}$ induces maps $\widetilde{\mathrm{A}}^I \to \widetilde{\mathrm{A}}^{pI}$ and $\widetilde{\mathrm{B}}^I \to \widetilde{\mathrm{B}}^{pI}$.

1.5. **The rings** $\mathrm{A}^+_{\max}$**,** $\mathrm{B}^+_{\max}$ **and** $\mathrm{B}_{\max}$**.** We set $r_n = p^{n-1}(p-1)$ so that $r_0 = \frac{p-1}{p}$. Then we define

$$\mathrm{B}^+_{\max} = \widetilde{\mathrm{B}}^{[0,r_0]} = \widetilde{\mathrm{B}}^{\left[0,\frac{p-1}{p}\right]} = \mathrm{A}_{\inf}\left\langle \frac{\left[\varpi^{\frac{p-1}{p}}\right]}{p}\right\rangle[1/p] = \mathrm{A}_{\inf}\left\langle\frac{[p^\flat]}{p}\right\rangle[1/p]$$

and similarly $\mathrm{A}^+_{\max} = \widetilde{\mathrm{A}}^{[0,r_0]}$. The ring $\mathrm{B}^+_{\max}$ admits a map to $\mathrm{B}_{\mathrm{dR}^+}$. Indeed, we have the identity

$$\frac{[p^\flat]}{p} = 1 + \left(\frac{[p^\flat]-p}{p}\right) \equiv 1 \mod \ker\theta,$$

so sequences $\sum_n a_n \left(\frac{[p^\flat]}{p}\right)^n$ with $a_n \to 0$ in the $\left(p, [p^\flat]\right)$-topology also have $a_n\left(\frac{[p^\flat]}{p}\right)^n \to 0$ in the $\ker\theta$-topology.

The rings $\mathrm{A}^+_{\max}$, $\mathrm{B}^+_{\max}$ and $\mathrm{B}_{\max}$ have an action of φ via the composition $\widetilde{\mathrm{B}}^{[0,r_0]} \xrightarrow{\varphi} \widetilde{\mathrm{B}}^{[0,pr_0]} \hookrightarrow \widetilde{\mathrm{B}}^{[0,r_0]}$.

The element $t = \log\left[1^\flat\right]$ makes sense in $\mathrm{B}^+_{\max}$. Indeed, we have $[1^\flat]-1 \in \ker\theta = \left(p - [p^\flat]\right)$, so that $\frac{\left([1^\flat]-1\right)^n}{n} = (\text{integral})\,\frac{p^n\left(\frac{[p^\flat]}{p}\right)}{n} \to 0$ in $\mathrm{A}_{\inf}\left\langle\frac{[p^\flat]}{p}\right\rangle$. It follows that

$$t = \log\left[1^\flat\right] = \sum_{n\geq 1}(-1)^{n-1}\,\frac{\left([1^\flat]-1\right)^n}{n}$$

converges in $\mathrm{A}_{\inf}\left\langle\frac{[p^\flat]}{p}\right\rangle[1/p]$.

1.6. **The rings** $\mathrm{A}^+_{\mathrm{cris}}$**,** $\mathrm{B}^+_{\mathrm{cris}}$ **and** $\mathrm{B}_{\mathrm{cris}}$**.** The ring $\mathrm{A}^+_{\mathrm{cris}}$ is defined to be the p-adic completion of $\mathrm{A}_{\inf}\left[\frac{\left(p-[p^\flat]\right)^n}{n!}\right]_{n\geq 1}$. We define $\mathrm{B}^+_{\mathrm{cris}} = \mathrm{A}^+_{\mathrm{cris}}[1/p]$. Since $[1^\flat]-1 \in \ker\theta = \left(p-[p^\flat]\right)$, we have

$$\frac{\left([1^\flat]-1\right)^n}{n} = (n-1)!\,(\text{integral})\,\frac{\left(p-[p^\flat]\right)^n}{n!}$$

so that $t = \log\left[1^\flat\right]$ belongs to $\mathrm{B}^+_{\mathrm{cris}}$. We set $\mathrm{B}_{\mathrm{cris}} = \mathrm{B}^+_{\mathrm{cris}}[1/t]$.

The ring $\mathrm{B}^+_{\mathrm{cris}}$ is very close to being equal to $\widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}\right]} = \mathrm{A}_{\inf}\left\langle\frac{[p^\flat]^{(p-1)}}{p}\right\rangle[1/p]$. On the one hand, we claim there is a containment $\mathrm{B}^+_{\mathrm{cris}} \subset \widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}\right]}$. Indeed, given n we will show that $\frac{\left(p-[p^\flat]\right)^n}{n!} \in \widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}\right]}$. Write $n = k(p-1)+r$ with $0 \leq r \leq p-2$, and remember that $v_p(n!) = \frac{n-s_p(n)}{p-1} = k + \frac{r-s_p(n)}{p-1}$, where $s_p(n)$ is the sum of digits of n in base p. Then we have

$$\frac{\left(p-[p^\flat]\right)^n}{n!} = (\text{unit})\left(\frac{\left(p-[p^\flat]\right)^{p-1}}{p}\right)^k p^{\frac{s_p(n)-r}{p-1}}\left(p-[p^\flat]\right)^r.$$

Now, we see that $\left(\frac{(p-[p^\flat])^{p-1}}{p}\right)^k$ belongs to $\widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}\right]}$, as well as $\left(p-[p^\flat]\right)^r$. On the other hand $\frac{s_p(n)-r}{p-1} \geq \frac{-p-2}{p-1} > -1$, but it is also an integer, so $\frac{s_p(n)-r}{p-1} \geq 0$. It follows that $p^{\frac{s_p(n)-r}{p-1}}$ is a positive power of p, so we see that the entire product, which is $\frac{(p-[p^\flat])^n}{n!}$, lies in $\widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}\right]}$. It is p-adically complete, so $\mathrm{B}^+_{\mathrm{cris}} \subset \widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}\right]}$.

In the other direction, it is almost true that $\widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}+r\right]} = \mathrm{A}_{\mathrm{inf}}\left\langle \frac{[p^\flat]^{(p-1)+pr}}{p}\right\rangle[1/p] \subset \mathrm{B}^+_{\mathrm{cris}}$. However there are some rationality problems, i.e. $\mathrm{B}^+_{\mathrm{cris}}$ only contains some power of the variable $\frac{[p^\flat]^{(p-1)+pr}}{p}$. Indeed, give such an r, take n divisible by $p-1$ and such that $s_p((p-1)\,n) \leq prn$. This is possible because s_p grows logarithmically. We then have

$$\frac{\left(p-[p^\flat]\right)^{(p-1)n}}{((p-1)n)!} = p^{s_p(n)}\left(\frac{\left(p-[p^\flat]\right)^{p-1}}{p}\right)^n \equiv_{\ \mathrm{mod}\ \mathrm{A}_{\mathrm{inf}}} p^{s_p(n)}\left(\frac{[p^\flat]^{p-1}}{p}\right)^n.$$

This shows that $p^{s_p(n)}\left(\frac{[p^\flat]^{p-1}}{p}\right)^n \in \mathrm{B}^+_{\mathrm{cris}}$. This element divides $p^{prn}\left(\frac{[p^\flat]^{p-1}}{p}\right)^n = \left(\frac{[p^\flat]^{(p-1)+pr}}{p}\right)^n$, so we see that $\left(\frac{[p^\flat]^{(p-1)+pr}}{p}\right)^n \in \mathrm{B}^+_{\mathrm{cris}}$ as claimed. Making this argument a little more precise will also show that $\left(\frac{[p^\flat]^{(p-1)+pr}}{p}\right)^n \in \mathrm{B}^+_{\mathrm{cris}}$ for all $n >> 0$.

We summarize the above discussion in the following proposition.

Proposition 1.1. *We have* $\mathrm{B}^+_{\mathrm{cris}} \subset \widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}\right]}$. *For any* $r > 0$ *there exists a finite map* $\widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}+r\right]} \to \mathrm{B}_r =$ *such that* $\mathrm{B}_r \subset \mathrm{B}^+_{\mathrm{cris}}$.

We notice a few more things. The first is that $\widetilde{\mathrm{B}}^{[0,p-1]} \subset \mathrm{B}^+_{\mathrm{cris}}$ (no need for a finite extension). The reason is that the coordinate of $\widetilde{\mathrm{B}}^{[0,p-1]}$ is given by $\frac{[p^\flat]^p}{p}$, which is equivalent mod $\mathrm{A}_{\mathrm{inf}}$ to a unit times $\frac{(p-[p^\flat])^p}{p!}$. It then follows from some algebra of divided powers that all powers of $\frac{[p^\flat]^p}{p}$ lie in $\mathrm{B}^+_{\mathrm{cris}}$. The ring $\mathrm{B}^+_{\mathrm{cris}}$ is endowed with a Frobenius map, because of an identity having to do with divided powers. Moreover, we have

$$\varphi\left(\mathrm{B}^+_{\mathrm{max}}\right) = \widetilde{\mathrm{B}}^{[0,p-1]} \subset \mathrm{B}^+_{\mathrm{cris}} \subset \widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)^2}{p}\right]} \subset \widetilde{\mathrm{B}}^{\left[0,\frac{(p-1)}{p}\right]} = \mathrm{B}^+_{\mathrm{max}},$$

so another way we can think of φ is by being the map induced from $\mathrm{B}^+_{\mathrm{cris}} \hookrightarrow \mathrm{B}^+_{\mathrm{max}} \xrightarrow{\varphi} \varphi(\mathrm{B}^+_{\mathrm{max}}) \hookrightarrow \mathrm{B}^+_{\mathrm{cris}}$.

1.7. **The rings $\widetilde{\mathrm{B}}^+_{\mathrm{rig}}, \widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$ and $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}$.** We set $\widetilde{\mathrm{B}}^+_{\mathrm{rig}} = \widetilde{\mathrm{B}}^{[0,\infty)}$, $\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}} = \widetilde{\mathrm{B}}^{[r,\infty)}$ and $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}} = \cup_{r\geq 0}\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$. The Frobenius map induces $\varphi : \widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}} \to \widetilde{\mathrm{B}}^{\dagger,pr}_{\mathrm{rig}}$ and hence $\varphi : \widetilde{\mathrm{B}}^+_{\mathrm{rig}} \to \widetilde{\mathrm{B}}^+_{\mathrm{rig}}$ and $\varphi : \widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}} \to \widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}$.

It is useful to note that $\widetilde{\mathrm{B}}_{\mathrm{rig}}^{+}=\cap_{n\geq 1}\varphi^{-n}\left(\mathrm{B}_{\max}^{+}\right)=\cap_{n\geq 1}\varphi^{-n}\left(\mathrm{B}_{\mathrm{cris}}^{+}\right)$, which makes φ into an automorphism of $\widetilde{\mathrm{B}}_{\mathrm{rig}}^{+}$.

2. The geometric spaces

The spaces we shall work with here are adic or pre-adic spaces. The specfic formalism of adic spaces is not too important for us, but if (R,R^{+}) is a Huber pair, $\mathrm{Spa}(R,R^{+})$ is a space whose points correspond to valuations and whose functions are basically R. The space $\mathrm{Spa}(R,R^{+})$ is not always a locally ringed space (the structure presheaf is not always a sheaf), and that's the distinction between being a pre-adic and an adic space. If x is a point, we denote the valuation by $f\mapsto |f(x)|$. There is a well defined operation of evaluating at a point: if x is a point, then the kernel of its valuation is a prime ideal of R, and so we think of $\mathrm{Frac}\left(R/\ker|\cdot|\right)$ as being the residue field at x. Finally, we note that these valuations may be valued in strange groups, but the operation of "maximal generization" always returns a point whose valuation is valued in $\mathbb{R}_{\geq 0}$ (see section 4.2 of the Berkeley notes). Recall also an analytic point is a valuation whose kernel is nonopen.

Let $\mathrm{A}_{\mathrm{inf}}$ be as in section 1. The space $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}})=\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}},\mathrm{A}_{\mathrm{inf}})$ is a pre-adic space. It is probably also an adic space but it's not clear if that is known at the moment (see footnote in section 12 of Berkeley notes). In $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}})$, it is useful to denote four special points by their residue fields.

1. $x_{\mathbb{F}_p}$ is the unique non-analytic point, given by $\mathrm{A}_{\mathrm{inf}}\twoheadrightarrow\mathrm{A}_{\mathrm{inf}}/\left(p,\left[p^{\flat}\right]\right)=\mathbb{F}_p$.
2. $x_{\mathbb{C}_p^{\flat}}$ is given by $\mathrm{A}_{\mathrm{inf}}\twoheadrightarrow\mathrm{A}_{\mathrm{inf}}/p=\mathcal{O}_{\mathbb{C}_p^{\flat}}\to\mathbb{C}_p^{\flat}$.
3. $x_{\mathbb{C}_p}$ is given by $\mathrm{A}_{\mathrm{inf}}\twoheadrightarrow\mathrm{A}_{\mathrm{inf}}/\left(p-\left[p^{\flat}\right]\right)=\mathcal{O}_{\mathbb{C}_p}\to\mathbb{C}_p$.
4. $x_{\mathbb{Q}_p}$ is given by $\mathrm{A}_{\mathrm{inf}}\twoheadrightarrow\mathrm{A}_{\mathrm{inf}}/\left[p^{\flat}\right]=\mathbb{Z}_p\to\mathbb{Q}_p$.

This is basically how all points look like, at least those which correspond to close prime ideals. See Colmez's survey on the Fargues-Fontaine curve, corollary 3.3.

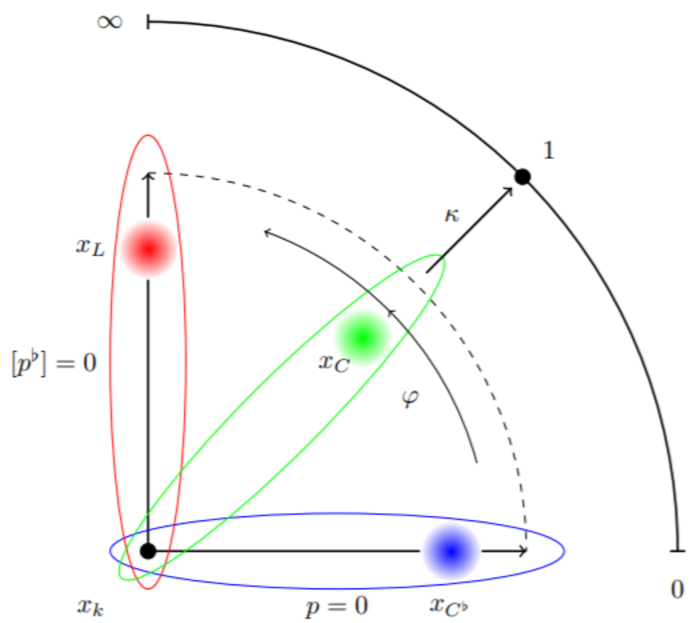
We let $\mathcal{Y}=\mathrm{Spa}\mathrm{A}_{\mathrm{inf}}-\left\{x_{\mathbb{F}_p}\right\}$, which is known to be an analytic adic space. There exists a surjective continuous map $\kappa:\mathcal{Y}\to[0,\infty]$, given by

$$\kappa(x)=\frac{\log\left|\left[p^{\flat}\right]\left(\widetilde{x}\right)\right|}{\log\left|p\left(\widetilde{x}\right)\right|}.$$

We have $\kappa\circ\varphi=p\kappa$. In particular,

$$\kappa\left(x_{\mathbb{C}_p^{\flat}}\right)=0,\kappa\left(x_{\mathbb{C}_p}\right)=1,\kappa\left(\varphi^{n}\left(x_{\mathbb{C}_p}\right)\right)=p^{n},\kappa\left(x_{\mathbb{C}_p^{\flat}}\right)=\infty.$$

This can be seen in the following picture, from page 101 of the Scholze-Weinstein notes.



For an interval $I \subset [0, \infty]$, we let $\mathcal{Y}_I$ be the interior of the preimage of $\mathcal{Y}$ under κ.

We visualize $\mathcal{Y}$ as being a sphere, and $\kappa = 0$ and $\kappa = \infty$ correspond to two opposing poles of this sphere. In the middle $\kappa = r$ represent circles lying in between.

With this in place, we may now give a geometric interpertation for the various rings appearing above.

2.1. **The ring $\mathbb{C}_p^\flat$.** The ring $\mathbb{C}_p^\flat$ is the residue ring of $x_{\mathbb{C}_p^\flat}$ in $\mathcal{Y}$. Thus $\mathbb{C}_p^\flat = k\left(x_{\mathbb{C}_p^\flat}\right)$.

2.2. **The ring $\mathrm{A}_{\mathrm{inf}}$.** The ring $\mathrm{A}_{\mathrm{inf}}$ is the coordinate ring of $\mathrm{Spa}\mathrm{A}_{\mathrm{inf}}$.

2.3. **The rings $\mathrm{B}_{\mathrm{dR}}^+$ and B_{dR}.** The ring $\mathrm{B}_{\mathrm{dR}}^+$ is the completion of the local ring of $x_{\mathbb{C}_p}$. Thus $\mathrm{B}_{\mathrm{dR}}^+ = \widehat{\mathcal{O}}_{\mathcal{Y}, x_{\mathbb{C}_p}}$. We may think of $t = \log\left[1^\flat\right]$ as giving a choice of a local coordinate. Thus if we are given a function which is defined in $x_{\mathbb{C}_p}$, it has an image in $\mathrm{B}_{\mathrm{dR}}^+$, and this image is its Taylor expansion at $x_{\mathbb{C}_p}$ in terms of t. Evaluating this function gives an element of $\mathbb{C}_p$, which is the same as taking the image through the homomorphism $\mathrm{B}_{\mathrm{dR}}^+ \to \mathrm{B}_{\mathrm{dR}}^+/t$.

2.4. **The ring $\widetilde{\mathrm{B}}^I$.** Let $\rho(r) := \frac{p-1}{pr}$ and $\rho(\infty) = 0$. This operation reverses directions between r and s and renormalizes. If $I = [r, s]$, let $\rho(I) = [\rho(s), \rho(r)]$. The ring $\widetilde{\mathrm{B}}^I$ is none other than $\mathrm{H}^0\left(\mathcal{Y}_{\frac{p}{p-1}I}, \mathcal{O}_{\mathcal{Y}}\right)$, the rings of functions converging on a closed rational set. In fact if I is closed (maybe also need $0 \notin I$), this set is an affinoid (proved by Kedlaya and Liu), so can be thought of as a coordinate ring.

Since we think of $\mathcal{Y}$ as a sphere, if $I \subset (0, \infty)$ it's useful to think of this as being the the ring of functions converging on some annuli.

2.5. **The ring** $\mathrm{B}^+_{\max}$**.** We have $\rho\left([0, \frac{p-1}{p}]\right) = [1, \infty]$, so $\mathrm{B}^+_{\max}$ is the same as $\mathrm{H}^0\left(\mathcal{Y}_{[1,\infty]}, \mathcal{O}_{\mathcal{Y}}\right)$. Thus it is the ring of functions on a closed disc.

2.6. **The ring** $\mathrm{B}^+_{\mathrm{cris}}$**.** It follows from the discussion before that $\mathrm{B}^+_{\mathrm{cris}}$ is something like the ring of functions on a space, which on the one hand covers the disc $\mathcal{Y}_{[\frac{1}{p-1},\infty]}$, and on the other hand for $r > 0$ admits cover by a finite covering of the slightly larger disc $\mathcal{Y}_{[r,\infty]}$ for $r < \frac{1}{p-1}$.

2.7. **The rings** $\widetilde{\mathrm{B}}^+_{\mathrm{rig}}, \widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$ **and** $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}$**.** The ring $\widetilde{\mathrm{B}}^+_{\mathrm{rig}}$ is the ring of functions $\mathrm{H}^0\left(\mathcal{Y}_{(0,\infty]}, \mathcal{O}_{\mathcal{Y}}\right)$, the ring $\widetilde{\mathrm{B}}^{\dagger,r}_{\mathrm{rig}}$ is the ring of functions $\mathrm{H}^0\left(\mathcal{Y}_{(0,\frac{p-1}{pr}]}, \mathcal{O}_{\mathcal{Y}}\right)$ (on a punctured annulus), and $\widetilde{\mathrm{B}}^{\dagger}_{\mathrm{rig}}$ is the local ring on at the puncture of 0, in other words it's the local ring $\mathcal{O}_{\mathcal{Y},x_{\mathbb{C}_p^\flat}}$ except that we also allows arbitrary poles at $x_{\mathbb{C}_p^\flat}$.