FIBERED CATEGORIES AND STACKS

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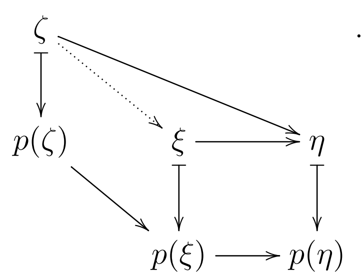
These notes follow these of Vistoli but add some examples.

1. Fibered categories

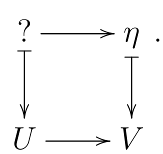
Fix categories $\mathcal{F}$ and $\mathcal{C}$ and a functor $p : \mathcal{F} \to \mathcal{C}$. We think of $\mathcal{F}$ as lying above $\mathcal{C}$.

A morphism $\xi \to \eta$ in $\mathcal{F}$ is Cartesian if for each $\zeta \to \eta$ we can solve uniquely



If $U \to V$ is given in $\mathcal{C}$ and $\xi \to \eta$ is Cartesian above $U \to V$, we say ξ is giving a pullback of η to U. It follows from the definitions a pullback is unique up to unique isomorphism.

Definition 1.1. We say $\mathcal{F}$ is fibered over $\mathcal{C}$ if we can pullback objects of $\mathcal{F}$ along any arrow of $\mathcal{C}$. In other words, we can always solve



A morphism of fibered categories $F : \mathcal{F} \to \mathcal{G}$ is a functor satsifying

(i) $p_{\mathcal{G}} \circ F = p_{\mathcal{F}}$ (an equality on the nose)

(ii) Cartesian $\mapsto$ Cartesian.

Now assume $\mathcal{F}$ is fibered. For an object $U \in \mathcal{C}$, the fiber $\mathcal{F}(U)$ is defined to be the subcategory of $\mathcal{F}$ of objects $\xi \mapsto U$, together with morphisms $(\xi_1 \to \xi_2) \mapsto \left(U \xrightarrow{\mathrm{Id}} U\right)$.

As we shall see in a bit, the point of this terminology is to be able to make sense of sheaves of categories. For a fibered category $p : \mathcal{F} \to \mathcal{C}$ it is close to being true that $U \mapsto \mathcal{F}(U)$ defines a presheaf. More on this in the next section.

2. Pseudo-functors

A pseudo-functor is a generalization of the concept a functor $\mathcal{C} \to$ Categories, which allows composition of morphisms to be equal only up to a natural isomorphism.

Definition 2.1. A pseudo-functor on a category $\mathcal{C}$ is the following data.

1. For each object U in $\mathcal{C}$, a category $\mathcal{F}(U)$.
2. For each morphism $f : U \to V$ in $\mathcal{C}$, a functor $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$.
3. For each object U in $\mathcal{C}$, a natural isomorphism $\varepsilon_U : \mathrm{Id}_U^* \xrightarrow{\sim} \mathrm{Id}_{\mathcal{F}(U)}$ of the two functors $\mathrm{Id}_U^*, \mathrm{Id}_{\mathcal{F}(U)} : \mathcal{F}(U) \to \mathcal{F}(U)$.
4. For each pair of morphisms $U \xrightarrow{f} V \xrightarrow{g} W$ a natural isomorphism of functors $\alpha_{f,g} : f^*g^* \xrightarrow{\sim} (gf)^*$.

These are required to satisfy the following conditions:

(a) If $U \xrightarrow{f} V$ is a morphism in $\mathcal{C}$, then $\alpha_{\mathrm{Id}_U,f} = \varepsilon_U \circ f^*$ and $\alpha_{f,\mathrm{Id}_V} = f^* \circ \varepsilon_V$, both equalities of natural isomorphisms.

(b) Whenever we have $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$ and an object η of $\mathcal{F}(T)$, the diagram

$$\begin{array}{ccc} f^*g^*h^*\eta & \xrightarrow{\alpha_{f,g}(h^*\eta)} & (gf)^*h^*\eta \\ \big\downarrow{\scriptstyle f^*\alpha_{g,h}(\eta)} & & \big\downarrow{\scriptstyle \alpha_{gf,h}(\eta)} \\ f^*(hg)^*\eta & \xrightarrow{\alpha_{f,hg}(\eta)} & (hgf)^*\eta \end{array}$$

commutes.

The idea is that we replace the *conditions* of a functor that the identity morphism maps to the identity morphism and that composition maps to composition, by adding *extra structure* of natural isomorphisms replacing the equalities. These new structures are required to agree to one higher order.

To make the connection with fibered categories, we need to make one more definition.

Definition 2.2. A cleavage of a fibered category $\mathcal{F} \to \mathcal{C}$ consists of a class K of cartesian arrows, a unique one for each diagram

$$\begin{array}{ccc} ? & \longrightarrow & \eta \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{f} & V \end{array}.$$

which we call $f^*\eta = f_K^*\eta$.

Clearly, a cleavage always exists by the axiom of choice. (Note that pullback are unique up to unique isomorphism, but still we wish to make to specify the different choices even though they only differ by a unique isomorphism).

Theorem 2.1. *There is a canonical 1-1 correspondence*

$$\{\textit{fibered categories over } \mathcal{C} \textit{ with a choice of cleavage}\} / \{\textit{isomorphism}\} \leftrightarrow$$
$$\{\textit{pseudo functor } \mathcal{C}^{\mathrm{op}} \to \textit{Categories}\} / \{\textit{isomorphism}\}.$$

To get a pseudo functor from a fibered category with a cleavage, we map

$$(p : \mathcal{F} \to \mathcal{C}, K) \mapsto \left(U \mapsto \mathcal{F}, \left(U \xrightarrow{f} V\right) \mapsto \left(\mathcal{F}(V) \xrightarrow{f_K^*} \mathcal{F}(U)\right)\right).$$

In the other direction, given $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to$ Categories, we denote by abuse of notation by $\mathcal{F}$ the category over $\mathcal{C}$, for which:

1. The objects are pairs (ξ, U) where $U \in \mathcal{C}$ and $\xi \in \mathcal{F}(U)$.

2. The morphisms between (ξ, U) and (η, V) are pairs of morphisms $(\xi \to f^*\eta, U \xrightarrow{f} V)$ (recall $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$ is part of the data of $\mathcal{F}$).

Note that we really mean this 1-1 correspondence is defined on the isomorphism classes and not just on the equivalence classes. Also, it is apperant from the construction that under this correspondence, functors $\mathcal{C}^{\mathrm{op}} \to$ Sets correspond to categories fibered in sets (i.e. each fiber $\mathcal{F}(U)$ is not only a category but actually a set). In other words, persheaves in sets are in 1-1 correspondence with categories fibered in sets (in this case there is no need to choose a cleavage because there are non nontrivial isomorphisms)

3. Examples of fibered categories

1. **The fibered category of arrows.** If $\mathcal{C}$ is a category which has fibered products, let Arrow($\mathcal{C}$) be the category whose object are morphisms $U \xrightarrow{f} V$ and whose morphisms are commutative diagrams between two morphisms. Let Arrow($\mathcal{C}$) be the functor mapping $\left(U \xrightarrow{f} V\right) \mapsto V$. Then the condition of being a fibered category precisely translates to $\mathcal{C}$ having fiber products. The associated pseudo-functor maps V to the category of morphisms Arrow(V) of morphisms $U \xrightarrow{f} V$, i.e. morphisms with codomain V.

2. **The fibered category of sheaves.** Let $\mathcal{C}$ be a site. By a sheaf on an object X we mean a sheaf on the site $\mathcal{C}/X$, the comma category of X. This category is denoted $\mathrm{Sh}(X)$. Mapping $X \mapsto \mathrm{Sh}(X)$ gives a pseudo-functor, which corresponds to the fibered category $\mathrm{Sh}(\mathcal{C})/\mathcal{C}$ whose fiber over X is $\mathrm{Sh}(X)$.

3. **The fibered category of quasi-coherent sheaves.** Let $\mathcal{C} = \mathrm{Sch}/S$, where S is a scheme. Then $U \mapsto$ Qcoh(U) (the category of quasi-coherent sheaves on U) is a pseudo-functor, which corresponds to a fibered category $\mathrm{Qcoh}(\mathcal{C})/\mathcal{C}$.

4. **The fibered category associated to a representable functor**. Suppose $\mathcal{C}$ is a category and $X \in \mathcal{C}$ is an object. Then X gives rise to a functor $h_X : \mathcal{C}^{\mathrm{op}} \to$ Sets via $h_X(U) = \mathrm{Hom}(U, X)$. This corresponds to a fibered category $\mathcal{C}_X/\mathcal{C}$. The objects of $\mathcal{C}_X$ are morphisms $U \to X$ with $U \in \mathcal{C}$. There are no nontrivial morphisms. The functor $\mathcal{C}_X \to \mathcal{C}$ maps $U \to X$ to U. The moral is that any object of a category $\mathcal{C}$ gives rise to a category fibered in sets above $\mathcal{C}$.

(**Warning:** a functor from $\mathcal{C}^{\mathrm{op}} \to$ Sets doesn't have to appear in this way).

5. **The modular stack.** Let $\mathcal{C} = \mathrm{Sch}$ be the category of schemes. The modular stack (the terminology will be justified later) is the fibered category Ell $\to$ Sch whose objects are elliptic

curves $E \to S$ over a base scheme S and whose morphisms are cartesian diagrams

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}.$$

The functor Ell $\to$ Sch maps $E \to S$ to S. Given such an S, the fiber $\mathrm{Ell}(S)$ is non other than the groupoid of elliptic curves over S. The corresponding psuedo-functor is the one mapping a Scheme S to the groupoind of elliptic curves $\mathrm{Ell}(S)$.

6. **Ell_N for $N \geq 5$.** Now suppose instead we consider Ell_N for $N \geq 5$, the fibered category of elliptic curves together with a trivialization of their N-torsion. Because there are no nontrivial automorphisms of an elliptic curve preserving its N-torsion, it follows that $\mathrm{Ell}_N \to \mathrm{Sch}$ is a catgeory fibered in sets. In fact, it turns out this is a special case of example 4. There is an object $X(N) \in \mathrm{Sch}$ such that the fibered categories $\mathrm{Ell}_N/\mathrm{Sch}$ and $\mathrm{Sch}_{X(N)}/\mathrm{Sch}$ are isomorphic. This just means that the sets $\mathrm{Ell}_N(S)$ and $X(N)(S)$ are naturally isomorphic.

4. Stacks

Let $\mathcal{C}$ be a site. Morally, a stack is a sheaf over $\mathcal{C}$. To make this precise, we need the following. Let $\mathcal{F}$ be a fibered category and fix a cleavage. Given a covering $\{U_i \to U\}$, let $U_{ij} = U_i \times_U U_j$ and $U_{ijk} = U_i \times_U U_j \times_U U_k$.

Definition 4.1. Let $\{U_i \to U\}$ be a covering. An object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ on $\{U_i \to U\}$ is a collection of objects $\xi_i \in \mathcal{F}(U_i)$ with isomorphisms $\phi_{ij} : \mathrm{Pr}_2^* \xi_j \xrightarrow{\sim} \mathrm{Pr}_1^* \xi_i$ in $\mathcal{F}(U_i \times_U U_j)$, such that

$$\mathrm{Pr}_{13}^* \phi_{ik} = \mathrm{Pr}_{12}^* \phi_{ij} \circ \mathrm{Pr}_{23}^* \phi_{jk}$$

as morphisms $\mathrm{Pr}_3^* \xi_k \to \mathrm{Pr}_1^* \xi_i$. A morphism between objects with descent data $\{\alpha_i\} : (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$ is a collection of morphisms $\alpha_i : \xi_i \to \eta_i$ with the property that for each pair i, j of indices, the diagram

$$\begin{array}{ccc} \mathrm{Pr}_2^* \xi_j & \xrightarrow{\mathrm{Pr}_2^* \alpha_j} & \mathrm{Pr}_2^* \eta_j \\ \downarrow \phi_{ij} & & \downarrow \psi_{ij} \\ \mathrm{Pr}_1^* \xi_i & \xrightarrow{\mathrm{Pr}_1^* \alpha_i} & \mathrm{Pr}_1^* \eta_i \end{array}.$$

The category of objects with descent data with respect to $\{U_i \to U\}$ is denoted $\mathcal{F}(\{U_i \to U\})$.

Given $\xi \in \mathcal{F}(U)$ we get the data of $\xi_i = \sigma_i^* \xi$ for $\sigma_i : U_i \to U$ and isomorphisms $\phi_{ij} : \mathrm{Pr}_2^* \xi_j \xrightarrow{\sim} \mathrm{Pr}_1^* \xi_i$ since $\mathrm{Pr}_2^* \xi_j = (\sigma_j \circ \mathrm{Pr}_2)^* \xi = (\sigma_i \circ \mathrm{Pr}_1)^* \xi = \mathrm{Pr}_1^* \xi_i$. Given a morphism $\xi \xrightarrow{\alpha} \eta$ in $\mathcal{F}(U)$ we get a a morphism of objects with descent data $\{\alpha_i\} : (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$ by setting $\alpha_i = \sigma_i^* \alpha$. Thus we obtain a functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$. This allows us to define what stacks are.

Definition 4.2. Let $\mathcal{F} \to \mathcal{C}$ be a fibered category and let $\mathcal{C}$ be a site.

(i) $\mathcal{F}$ is a prestack over $\mathcal{C}$ if for each covering $\{U_i \to U\}$ in $\mathcal{C}$ the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is fully faithful.

(ii) $\mathcal{F}$ is a stack over $\mathcal{C}$ if for each $\{U_i \to U\}$ in $\mathcal{C}$ the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is an equivalence of categories.

5. Examples of stacks

1. **Categories fibered in sets.** If $\mathcal{C}$ is a site and $\mathcal{F}/\mathcal{C}$ is fibered in sets, so that $\mathcal{F}$ corresponds to a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ then

(i) $\mathcal{F}$ is a prestack if and only if F is a separated presheaf.

(ii) $\mathcal{F}$ is a stack if and only if F is a sheaf.

In particular, if $\mathcal{C}$ is site in which representable objects define sheaves via the Yoneda embedding (aka subcanonical sites), then they also give rise to stacks. For example, in the Zariski, etale, fppf and fpqc topologies, schemes give rise to stacks through example 4 of §3.

2. **The fibered category of sheaves.** Let $\mathcal{C}$ be a site. By a sheaf on an object X we mean a sheaf of sets on the site $\mathcal{C}/X$, the comma category of X. This category is denoted $\mathrm{Sh}(X)$. Mapping $X \mapsto \mathrm{Sh}(X)$ gives a pseudo-functor, which corresponds to the fibered category $\mathrm{Sh}(\mathcal{C})/\mathcal{C}$ whose fiber over X is $\mathrm{Sh}(X)$. (There are standard things one needs to check. This is example 4.11 of Vistoli's notes).

3. **The fibered category of quasi-coherent sheaves.** Let $\mathcal{C} = \mathrm{Sch}/S$, where S is a scheme, endowed with the fpqc topology. Then $U \mapsto \mathrm{Qcoh(U)}$ (the category of quasi-coherent sheaves on U) is a pseudo-functor, which corresponds to a fibered category $\mathrm{Qcoh}(\mathcal{C})/\mathcal{C}$, is a stack.

Essentially, this comes down to it being a stack in the Zariski topology (which is obvious) and faithfully flat descent for affine schemes (this is lemma 4.25 of Vistoli, which is pretty useful). In this context, faithfully flat descent for affine schemes means that if $A \to B$ is a faithfully flat map of rings then Mod_A is equivalent to the category of pairs (N, ϕ) of $N \in \mathrm{Mod}_B$ and an isomorphism of $B \otimes_A B$-modules $\phi : N \otimes_A B \xrightarrow{\sim} B \otimes_A N$ required to satisfied the standard compatibilities. The faithfull flatness is essential in that the sequence $0 \to A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B$ is exact. See theorem 2.5 of the CRing project, chapter 15 for a nice treatment.

4. **All schemes over the etale site do not form a stack.** Take $\mathcal{C} = (\mathrm{Sch})_{\text{ét}}$, and Schemes the category of pairs of schemes $\{S \to T\}$, mapping a pair to the base scheme T. Then Schemes is not a stack. The point is that sometimes descent data is not effective in the etale world. You could have a finite free action of a group G on a scheme X such that X/G is not a scheme, though this action gives rise to an effective descent datum. Such an object is called an algebraic space. Thus algebraic spaces is an enlargment of the category of schemes which makes it so that $\mathrm{AlgSpaces}/\left(\mathrm{Sch}\right)_{\text{ét}}$ is a stack. Think of something like $\mathrm{A}^1_{\mathbb{C}}/\mathrm{lattice}$.

5. **The modular stack.** Take $\mathcal{C} = (\mathrm{Sch})_{\mathrm{fppf}}$, $\mathcal{F} = \mathrm{Ell}$ the category from example 5 of §3. Then it turns out that Ell is a stack, though this is not obvious. The hard part is to show something like (according to the criteria mentioned in example 3 above) that if $A \to B$ is faithfully flat, then give a pair $\left(E_B \to B, \phi : E_B \times_B^{\mathrm{Pr}_2} (B \otimes_A B) \xrightarrow{\sim} E_B \times_B^{\mathrm{Pr}_1} (B \otimes_A B)\right)$ one can produce an elliptic curve $E \to A$ giving rise to it. As explained in 4, in general it is not possible to glue schemes in this way, but this can be done for elliptic curves. This is achieved by relating the construction of elliptic curves to line bundles and then ultimately to descent

for quasi-coherent sheaves which is possible. See the note "Moduli stack of elliptic curves" by Meier and Ozornova. The note "Introduction to algebraic stacks" by Voight also gives a nice discussion.

For example, we think of the $\mathbb{C}$-points $\mathrm{Ell}(\mathbb{C})$ of Ell as being the (groupoid) line A_j^1 over $\mathbb{C}$ with automorphisms at each point. There is a map of stacks $\mathrm{Ell} \to \mathrm{Spec}\mathbb{Z}[j]$ which on the level of $\mathbb{C}$-points is the functor $\mathrm{Ell}(\mathbb{C}) \to \mathrm{A}_j^1$ is doing nothing on objects but is killing all the automorphisms.

Recall that one reason that Ell/iso by a scheme was the following argument: if it were represented by a scheme X, consider two non-isomorphic elliptic curves over $\mathbb{Q}$ which become isomorphic over $\overline{\mathbb{Q}}$. Then we get a contradiction because $X(\mathbb{Q}) \to X(\overline{\mathbb{Q}})$ has to be injective.

How does this get compared to what happens for Ell? How is this problem fixed, and in what sense is it fixed?

Well, the condition "$X(\mathbb{Q}) \to X(\overline{\mathbb{Q}})$ is injective" can be reinterpreted in the langauge of sheaves. As discussed before, any scheme X is gives rise to a sheaf h_X on the fppf site; in particular, considering the fppf covering $\mathrm{Spec}\overline{\mathbb{Q}} \to \mathrm{Spec}\mathbb{Q}$, one must have that $h_X(\mathrm{Spec}\mathbb{Q}) \to h_X\left(\mathrm{Spec}\overline{\mathbb{Q}}\right)$ is injective. In other words, the separatedenss condition of a sheaf fails for Ell/iso as a presheaf on the fppf site.

However, for Ell, this sheaf separateness condition is replaced by asking that the functor of groupoids $\mathrm{Ell}(\mathrm{Spec}\mathbb{Q}) \to \mathrm{Ell}\left(\left\{\mathrm{Spec}\overline{\mathbb{Q}} \to \mathrm{Spec}\mathbb{Q}\right\}\right)$ is fully faithful. This condition now holds. The point is the following. Let E_1, E_2 be these two nonisomorphic elliptic curves over $\mathbb{Q}$ which become isomorphic over $\mathbb{Q}$. Then $E_1 \times \overline{\mathbb{Q}} \cong E_2 \times \overline{\mathbb{Q}}$, but the image of E_1, E_2 in $\mathrm{Ell}\left(\left\{\mathrm{Spec}\overline{\mathbb{Q}} \to \mathrm{Spec}\mathbb{Q}\right\}\right)$ are not isomorphic, because the descent datums are different! This is exactly because the isomorphism $E_1 \times \overline{\mathbb{Q}} \cong E_2 \times \overline{\mathbb{Q}}$ is not defined over $\mathbb{Q}$. That's why we do not see a unique descent from $\overline{\mathbb{Q}}$ to $\mathbb{Q}$. When the specific isomorphisms are not remembered (i.e. when we work with Ell/iso instead of Ell) all possible descent datums are identified (all isomorphisms are an equality), so this difference is not detected.

Furthermore, as a special case of 1 and example 6 of §3, we see that Ell_N gives rise to a stack fibered in sets which has for each S an action of the group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mathrm{Ell}_N(S)$. The quotient fibered category (when defined in the correct way) let us remark gives $\mathrm{Ell}_N/\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \cong \mathrm{Ell}$.

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