EIGENFORMS AND TRIANGULINE REPRESENTATIONS

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Abstract. These are notes for a talk following a "path of least resistance" towards eigenforms and trianguline representations, including a discussion of a result due to the author.

1. Representations associated to eigenforms

Let p be a prime and $N \geq 1$ an integer coprime to p. Recall that if $f \in S_k(\Gamma_1(N), \mathbb{Q})$ is an eigenform with $f(q) = \sum_{n \geq 1} a_n q^n$, then there exists a unique Galois representation

$$\rho_f : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2(\mathbb{Q}_p)$$

characterized by the relation $\rho_f(\mathrm{Frob}_l) = a_l$ for $l \nmid Np$.

Write $G_{\mathbb{Q}_p} = \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$. In this talk, we will be interested in the p-adic representation $\rho_{f,p} = \rho_f|_{G_{\mathbb{Q}_p}} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{Q}_p)$. There are two cases which differ significantly:

In the first case, $p \nmid a_p$, in which case we say f is ordinary. In that case $\rho_{f,p}$ is reducible, and can be written as an extension

$$0 \to \mathbb{Q}_p\left(\chi_{\mathrm{cyc}}^{k-1} \cdot \mathrm{un}_1\right) \to \rho_{f,p} \to \mathbb{Q}_p(\mathrm{un}_2) \to 0$$

where un_1 and un_2 are certain unramified characters. Another way to say this is that there is a basis in which each $g \in G_{\mathbb{Q}_p}$ has

$$\mathrm{Mat}(g) = \begin{bmatrix} \chi_{\mathrm{cyc}}(g)^{k-1}\,\mathrm{un}_1(g) & * \\ 0 & \mathrm{un}_2(g) \end{bmatrix},$$

so that the matrix of every element is *upper triangular.*

In the second case, $p \mid a_p$, in which case we say f is nonordinary. In that case $\rho_{f,p}$ is absolutely irreducible.

Thus, in the ordinary case, one can analyze $\rho_{f,p}$ by studying two characters and an extension class between them, which in practice is not too difficult to compute. On the face of it, it seems like there is no similar way to do this for $\rho_{f,p}$ when f is nonordinary. However, it turns out that it is actually possible to do something similar, if we allow ourselves to work in a category slightly larger than that of p-adic representations of $G_{\mathbb{Q}_p}$! The point of this talk is to present such an approach. The category in which $\rho_{f,p}$ will become reducible is the category of (φ, Γ)-modules, which is presented in the next section.

2. (φ,Γ)-modules

To define what (φ,Γ)-modules are, we must first describe the ring over which they live. This is the Robba ring $\mathcal{R}$; it is the ring of power series

$$\mathcal{R}=\left\{f(X)=\sum_{n\in\mathbb{Z}}a_nX^n : a_n\in\mathbb{Q}_p, f(X) \text{ converges in some annulus } r<|X|\le 1\right\}.$$

There are two natural operations we can perform on an annuli $(r,1]$.

There is the operation mapping $\varphi : X\mapsto (1+X)^p-1$, which inflates the annuli $(r,1]$ to the larger annuli $(r^p,1]$ (at least, if r is sufficiently close to 1).

The group $\Gamma=\mathbb{Z}_p^\times$ acts on each annuli $(r,1]$ isometrically by mapping $X\mapsto(1+X)^a-1:=\sum_{n\ge 1}\binom{a}{n}X^n$ (one can visualize this as some sort of rotation of the annuli $(r,1]$).

These two actions then induce actions on $\mathcal{R}$: we have $\varphi(f(X))=f((1+X)^p-1)$ and $af(X)=f((1+X)^a-1)$. We define a (φ,Γ)-module to be a finite free $\mathcal{R}$-module with commuting φ,Γ actions.

Example 2.1. Take a character $\delta:\mathbb{Q}_p^\times\to\mathbb{Q}_p^\times$. We define a rank 1 (φ,Γ)-module $\mathcal{R}(\delta)=\mathcal{R}e_\delta$, where

$$\varphi(e_\delta):=\delta(p)e_\delta,$$

$$a(e_\delta)=\delta(a)e_\delta.$$

Actually, it turns out that every rank 1 (φ,Γ)-module is isomorphic to some $\mathcal{R}(\delta)$.

The relation between p-adic representations and (φ,Γ)-modules can be (coarsely) described by the following theorem, which follows by combining theorems due to Fontaine, Cherbonnier-Colmez and Kedlaya.

Theorem 2.1. *There exists a fully faithful embedding*

$$\mathrm{D}:\left\{p-\textit{adic representations of } G_{\mathbb{Q}_p}\right\}\hookrightarrow\left\{(\varphi,\Gamma)-\textit{modules over } \mathcal{R}\right\}.$$

Example 2.2. Given a character $\eta:G_{\mathbb{Q}_p}^{\mathrm{ab}}\to\mathbb{Q}_p^\times$ we have a rank 1 representation $\mathbb{Q}_p(\eta)$ of $G_{\mathbb{Q}_p}$. Local class field theory gives a natural map $\mathrm{Art}:\mathbb{Q}_p^\times\to G_{\mathbb{Q}_p}^{\mathrm{ab}}$, and it turns out that

$$\mathrm{D}(\mathbb{Q}_p(\eta))=\mathcal{R}(\eta\circ\mathrm{Art}).$$

Note that $(\eta\circ\mathrm{Art})(p)=\eta\left(\mathrm{Frob}_p^{-1}\right)$ is a unit, but for a general rank 1 (φ,Γ)-module $\mathcal{R}(\delta)$, the element $\delta(p)$ does not have to be a unit. Thus D is not essentially surjective on rank 1 objects. In fact, it turns out the essential image consists exactly of these $\mathcal{R}(\delta)$ for which $\delta(p)$ is a unit.

3. Trianguline representations

A (φ,Γ)-module is called trianguline if it is a successive extension of rank 1 objects. Equivalently, if there is some basis in which $\mathrm{Mat}(\varphi)$ and $\mathrm{Mat}(a)$ for $a\in\mathbb{Z}_p^\times$ are all upper triangular.

In rank 2, a (φ, Γ)-module trianguline if and only if it is reducible[1]. A representation V of $G_{\mathbb{Q}_p}$ is called trianguline if $\mathrm{D}(V)$ is trianguline.

Example 3.1. If V is of rank 2 and reducible, then V is trianguline. Indeed if we write

$$0 \to \mathbb{Q}_p(\eta_1) \to V \to \mathbb{Q}_p(\eta_2) \to 0$$

for two characters η_1, η_2, then we have

$$0 \to \mathcal{R}(\eta_1 \circ \mathrm{Art}) \to \mathrm{D}(V) \to \mathcal{R}(\eta_2 \circ \mathrm{Art}) \to 0.$$

(Here we have implicitly used the fact that D is an exact functor). In particular, if f is an ordinary eigenform as in §1, then $\rho_{f,p}$ is trianguline.

The example above is not very surprising. The following theorem, which in our setting follows from the work of Scholl, Berger and Colmez, is more surprising.

Theorem 3.1. *If $f \in S_k(\Gamma_1(N), \mathbb{Q})$ is any eigenform where $p \nmid N$, then $\rho_{f,p}$ is trianguline (whether f is ordinary or not).*

Example 3.2. Take $f = \Delta \in S_{12}(\Gamma_1(1), \mathbb{Q})$ to be the Ramanujan Delta function and $p < 11$. It is has q-expansion $\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24}$, and one can check by hand or by basic results of Hida theory that for all $p < 11$ the eigenform Δ is nonordinary. This implies that $\rho_{\Delta,p}$ is absolutely irreducible. Nevertheless, we can still write $\mathrm{D}(\rho_{\Delta,p})$ as an extension. To describe it, let λ be a root of the Hecke polynomial $T^2 - \tau(p)T + p^{11}$ of Δ, and let

$$\delta_1 : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times; p \mapsto \lambda, \mathbb{Z}_p^\times \ni a \mapsto a^{11} \in \mathbb{Z}_p^\times,$$

and

$$\delta_2 : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times; p \mapsto \lambda^{-1}, \mathbb{Z}_p^\times \ni a \mapsto 1 \in \mathbb{Z}_p^\times.$$

Then $\mathrm{D}(\rho_{\Delta,p})$ is an extension

$$0 \to \mathcal{R}(\delta_1) \to \mathrm{D}(\rho_{\Delta,p}) \to \mathcal{R}(\delta_2) \to 0.$$

The extension type can also be described. In fact, there is a unique nonsplit extension of $\mathcal{R}(\delta_1)$ by $\mathcal{R}(\delta_2)$ up to isomorphism, and $\mathrm{D}(\rho_{\Delta,p})$ is that extension.
Note that $\delta_1(p) = \lambda$ and $\delta_2(p) = \lambda^{-1}$ are not units, since Δ is nonordinary. Thus by Example 2.2 $\mathcal{R}(\delta_1)$ and $\mathcal{R}(\delta_2)$ are not in the essential image of D, which explains why this extension does not exist on the level of representations.

4. Lubin-Tate triangulation

One may try to generalize Theorem 3.1 to other contexts, and such generalizations have been carried out successfully in many cases. We wish to describe in this section one such generalization which has been carried out by the author. It is one which replaces eigenforms in the sense of §1 by Hilbert eigenforms. Recall that Hilbert eigenforms are automorphic forms associated to $\mathrm{GL}_{2/F}$, where F is a totally real number field of degree d over $\mathbb{Q}$. Such

[1]The implication "reducible ⇒trianguline" does not follow automatically from the definitions, because there could be rank 1 (φ, Γ)-submodules which are not saturated; the problem is that $\mathcal{R}$ is only a ring, not a field. However, it turns out that any (φ, Γ)-submodule has a unique saturation.

Hilbert eigenforms can be thought of as having weights $\kappa = [k_1, ..., k_d]$. If $f \in S_\kappa(\Gamma_1(N), F)$ is such an eigenform, and v is a place of F, then there is an associated representation

$$\rho_f : \mathrm{Gal}\left(\overline{F}/F\right) \to \mathrm{GL}_2(F_v).$$

Let us work again under the assumption of §1 that the level N is coprime to the place v, and consider the p-adic representation

$$\rho_{f,v} = \rho_f|_{G_{F_v}} : G_{F_v} \to \mathrm{GL}_2(F_v).$$

If f is ordinary at v, in a suitable sense, this representation is again reducible, and otherwise it is irreducible.

If v splits then $F_v = \mathbb{Q}_p$ and Theorem 3.1 holds in exactly the same sense.

However, if v does not split, things become more interesting. We replace the Robba ring $\mathcal{R}$ of §2 by the Lubin-Tate Robba ring $\mathcal{R}_{\mathrm{LT}}$ of power series with coefficients in F_v instead of $\mathbb{Q}_p$. Now $\mathcal{R}_{\mathrm{LT}}$ carries an action of the much larger group $\Gamma_{F_v} = \mathcal{O}_{F_v}^\times$ compared to the action of $\mathbb{Z}_p^\times$ on $\mathcal{R}$. This action comes from Lubin-Tate theory: an element $a \in \mathcal{O}_{F_v}^\times$ acts on the variable T by mapping it to a power series $[a](T) = aT + ...$ which comes from the multiplication on a Lubin-Tate formal group of F_v.

In this context, a theorem of Berger replaces Theorem 2.1:

Theorem 4.1. *There exists a fully faithfuly functor*

$$\mathrm{D}_{\mathrm{LT}} : \{\textit{Analytic } p - \textit{adic representations of } G_{F_v}\} \hookrightarrow \{(\varphi, \Gamma) - \textit{modules over } \mathcal{R}_{\mathrm{LT}}\}.$$

The "analytic condition" has to do with the behaviour of the Hodge-Tate weights of the representations away from the identity embedding of $F_v \hookrightarrow \overline{\mathbb{Q}}_p$, and holds automatically for any representation if $F_v = \mathbb{Q}_p$.

Using this functor, we can define a notion of a "Lubin-Tate trianguline" representation, and ask whether or not any $\rho_{f,v}$ is Lubin-Tate trianguline. The answer turns out to be yes. For simplicity, let us state the theorem in the case that p is inert, so that $[F_p : \mathbb{Q}_p] = [F : \mathbb{Q}]$.

Theorem 4.2. *If $\rho_{f,v}$ has weights $\kappa = [k, 1, ..., 1]$, then $\rho_{f,v}$ is analytic and Lubin-Tate trianguline.*

As explained above, Theorem 3.1 follows from the work of Scholl, Berger and Colmez. The main innovation in the proof of Theorem 4.2 is to redo the work of Berger in the Lubin-Tate context, which requires the use of different p-adic Hodge theory techniques compared to those Berger used. This is due to substantial differences when working with F_v instead of $\mathbb{Q}_p$.

5. Concluding remarks

We add a few remarks concerning what has been omitted from this talk for ease of exposition.

1. More generally, Theorem 3.1 holds for p-adic eigenforms of finite slope. This was proved by Kisin, and is actually one of the main motivations for proving triangulinity. All of the examples presented here were actually crystalline representations, which can often be studied more simply without using (φ, Γ)-modules, via the $\mathrm{D}_{\mathrm{cris}}$ functor.

2. Similarly, Theorem 4.2 holds for p-adic eigenforms of finite slope, provided that $\rho_{f,v}$ is analytic (or more generally a twist of an analytic representation). This analyticity is known

in the crystalline context due to some local-global compatibility result, and one certainly conjectures it to hold whenever f is classical (i.e. even if v divides the level).

3. Prior to this work there were already existing very general triangulinity results regarding finite slope Hilbert eigenforms, due to Kedlaya-Pottharst-Xiao and Liu. They are however triangulinity results of a different kind, where the Robba ring has $[F_v : \mathbb{Q}_p]$ variables in a suitable sense. In contrast our Robba ring $\mathcal{R}_{\mathrm{LT}}$ is the ring of functions defined over a 1-dimensional annulus. However their results hold for finite slope p-adic eigenforms of any weight. Our result is supposed to capture only these special eigenforms which are "analytic", i.e. satisfy a Cauchy-Riemann equation of some sort. Only for such eigenforms it is possible to cut the number of dimensions from $[F_v : \mathbb{Q}_p]$ to 1. It also us to work with only 1 slope, which is important in applications, see remark 4 below.

In fact, not only these results are similar to ours, but actually, these results are used to perform much of the heavy lifting in the proof of (the generalized version of) Theorem 4.2. One major step of our work is to show that both types of triangulinty are actually equivalent whenever $\rho_{f,p}$ (or more generally, any p-adic representation) is analytic.

4. One of the first applications of Theorem 3.1 for finite slope eigenforms was Kisin's proof that the Fontaine-Mazur conjecture holds for representations $\rho_{f,p}$ coming from a finite slope eigenform f. The other major ingredient needed for this result was a classicality criterion due to Coleman, which is currently unavailable in our context. If such a classicality criterion were known (and based on conjectures of Breuil, it seems likely such a criterion exists) one could conclude from our work similarly more cases of the Fontaine-Mazur conjecture.

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