COMPUTING THE (φ,Γ)-MODULE ATTACHED TO A CRYSTALLINE KUMMER EXTENSION

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ABSTRACT. These are notes for a talk which computes the (φ,Γ)-module attached to a crystalline Kummer extension.

1. Introduction

According to a theorem of Fontaine, Cherbonnier-Colmez and Kedlaya, there is a fully faithful functor

$$\mathrm{D}:\left\{\text{Representations } \rho:\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)\to \mathrm{GL}_n\left(\mathbb{Q}_p\right)\right\}\hookrightarrow\left\{(\varphi,\Gamma)\text{-modules over the Robba ring}\right\}.$$

However, given a p-adic representation V, it is not clear at first how to compute the associated (φ,Γ)-module $\mathrm{D}(V)$. The point of this talk is to explain how to do this in a very simple case where V is a crystalline Kummer extension, following a method of Berger.

2. The crystalline Kummer extension

Recall that we have the $\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$-equivariant Kummer sequence:

$$0\to\mu_{p^n}\left(\overline{\mathbb{Q}}_p\right)\to\overline{\mathbb{Q}}_p^\times\xrightarrow{p^n}\overline{\mathbb{Q}}_p^\times\to 0.$$

Taking Galois cohomology, this gives

$$\mathbb{Q}_p^\times\otimes_{\mathbb{Z}_p}\mathbb{Z}_p/p^n\xrightarrow{\sim}\mathrm{H}^1\left(\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right),\mu_{p^n}\right),$$

with the isomorphism being

$$\alpha\otimes 1\mapsto\xi_\alpha,$$

where ξ_α is determined by the identity

$$\zeta_{p^n}^{\xi_\alpha(g)}=\frac{g\left(\alpha^{1/p^n}\right)}{\alpha^{1/p^n}},g\in\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right).$$

Taking the limit and tensoring with $\mathbb{Q}_p$, we get the Kummer isomorphism

$$\mathrm{Ku}:\mathbb{Q}_p^\times\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\xrightarrow{\sim}\mathrm{H}^1\left(\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right),\mathbb{Q}_p(1)\right).$$

Given an element $x\in\mathbb{Q}_p$, write $x^\flat$ for a compatible sequence of p^n'th roots of x. For example, we shall write $1^\flat=(1,\zeta_p,\zeta_{p^2},...)$ (nonstandard notation).

With this notation, the isomorphism Ku can be described concisely as mapping $\alpha\otimes 1\mapsto\xi_\alpha$ where $\left(1^\flat\right)^{\xi_\alpha(g)}=\frac{g(\alpha^\flat)}{\alpha^\flat}$ for $g\in\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$. The crystalline Kummer extensions are these Kummer extensions which correspond to α being an integral unit, i.e.

$$\mathbb{Z}_p^\times\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\xrightarrow{\sim}\left\{\text{Crystalline Kummer extensions}\right\}.$$

In terms of representations, $\alpha \otimes 1$ corresponds to the representation

$$g \in \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right) \mapsto \begin{pmatrix} \chi_{\mathrm{cyc}}(g) & \xi_\alpha(g) \\ 0 & 1 \end{pmatrix}.$$

Let us write Ku_α for this representation. Since $\mathbb{Z}_p^\times \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = (1+p\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ we will always assume $\alpha \in 1 + p\mathbb{Z}_p$.

3. Filtered φ-modules

A filtered φ-module is a $\mathbb{Q}_p$-vector space D, together with a linear isomorphism $\varphi : D \xrightarrow{\sim} D$ and a separated, exhaustive decreasing filtration

$$... \supset \mathrm{Fil}^{-1} D \supset \mathrm{Fil}^0 D \supset \mathrm{Fil}^1 D \supset ...$$

There is a functor

$$\mathrm{D}_{\mathrm{cris}} : \left\{\text{Representations } \rho : \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right) \to \mathrm{GL}_n\left(\mathbb{Q}_p\right)\right\} \to \left\{\text{Filtered } \varphi\text{-modules}\right\}.$$

When restricted to crystalline representations, such as the crystalline Kummer extensions, this functor is actually a fully faithful embedding.

The functor $\mathrm{D}_{\mathrm{cris}}$ is defined as follows. There is a $\mathbb{Q}_p$-algebra called $\mathrm{B}_{\mathrm{cris}}$; this ring has several structures: it has an action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$, an action of φ and a filtration. We then set

$$\mathrm{D}_{\mathrm{cris}}(V) := \left(\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V\right)^{\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)}.$$

To keep this talk at a reasonable length, we won't define $\mathrm{B}_{\mathrm{cris}}$ here. Instead, let us point out a property of $\mathrm{B}_{\mathrm{cris}}$ that is not difficult to verify from the definition. In fact this property gives quite a bit of motivation for the definition of $\mathrm{B}_{\mathrm{cris}}$. It is as follows: given an element of $x \in 1 + p\mathbb{Z}_p$, there is a multiplicative element $\left[x^\flat\right] \in \mathrm{B}_{\mathrm{cris}}$ so that $\left[x^\flat y^\flat\right] = \left[x^\flat\right]\left[y^\flat\right]$ and an element $\log\left[x^\flat\right] \in \mathrm{B}_{\mathrm{cris}}$ satisfying the identities one expects log to satisfy.

For example, there is an element $t := \log\left[1^\flat\right]$, so that

$$g(t) = g\left(\log\left[1^\flat\right]\right) = \log\left[\left(1^\flat\right)^{\chi_{\mathrm{cyc}}(g)}\right] = \log\left[(1^\flat)\right]^{\chi_{\mathrm{cyc}}(g)} = \chi_{\mathrm{cyc}}(g)\log\left[1^\flat\right] = \chi_{\mathrm{cyc}}(g)t.$$

Now let us compute $\mathrm{D}_{\mathrm{cris}}(\mathrm{Ku}_\alpha)$. By definition,

$$\mathrm{D}_{\mathrm{cris}}(\mathrm{Ku}_\alpha) = \left(\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \mathrm{Ku}_\alpha\right)^{\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)}.$$

By definition, we have a basis e, f of the 2-dimensional representation Ku_α for which the action is given by

$$g(e) = \chi_{\mathrm{cyc}}(g)e, g(f) = f + \xi_\alpha(g)e.$$

We need to find a 2-dimensional $\mathbb{Q}_p$-subspace in $\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \mathrm{Ku}_\alpha$ fixed by $\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$. First, we notice that

$$g(t^{-1} \otimes e) = \chi_{\mathrm{cyc}}(g)^{-1} t^{-1} \otimes \chi_{\mathrm{cyc}}(g)\, e = t^{-1} \otimes e,$$

so $t^{-1} \otimes e$ is a first such vector. To find another fixed vector, it is natural guess there is one of the form $f + t^{-1}b \otimes e$ with $b \in \mathrm{B}_{\mathrm{cris}}$. Then for $f + t^{-1}b \otimes e$ to be fixed by $\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$ we need to have $g(b)-b=-\xi_\alpha(g)t$. Where can we find such an element? The trick is to take $\log[\cdot]$ of both sides of the identity $(1^\flat)^{\xi_\alpha(g)}=\frac{g(\alpha^\flat)}{\alpha^\flat}$:

$$g\left(\log\left[\alpha^\flat\right]\right)=\log\left[\alpha^\flat\right]+\xi_\alpha t,$$

so that $f+t^{-1}b\otimes e$ is fixed for $b=-\log\left[\alpha^\flat\right]$.

Summary: writing $x=t^{-1}\otimes e$ and $y=f-\log\left[\alpha^\flat\right]t^{-1}\otimes e$ we have $\mathrm{D}_\alpha:=\mathrm{D}_{\mathrm{cris}}\left(\mathrm{Ku}_\alpha\right)=\mathbb{Q}_p\left(x\right)\oplus\mathbb{Q}_p\left(y\right)$, with $\varphi(x)=p^{-1}x$ and $\varphi(y)=y$. The filtration is determined by how many times elememts are divisible by t, and since $\log\left[\alpha^\flat\right]\equiv\log_p\alpha \bmod t$, we have

$$\mathrm{Fil}^{-1}\mathrm{D}_\alpha=\mathrm{D}_\alpha,\supset\mathrm{Fil}^0\mathrm{D}_\alpha=\mathbb{Q}_p\left(x\log_p\alpha+y\right)\supset\mathrm{Fil}^1\mathrm{D}_\alpha=0.$$

4. (φ,Γ)-MODULES

Recall the Robba ring $\mathrm{B}^\dagger_{\mathrm{rig}}$ is the ring of power series

$$\mathrm{B}^\dagger_{\mathrm{rig}}=\left\{f(X)=\sum_{n\in\mathbb{Z}}a_nX^n:a_n\in\mathbb{Q}_p,f(X)\text{ converges in some annulus }r<|X|\leq 1\right\}.$$

There are two natural operations we can perform on an annuli $(r,1]$. There is the operation mapping $\varphi:X\mapsto(1+X)^p-1$, which inflates the annuli $(r,1]$ to the larger annuli $(r^p,1]$ (at least, if r is sufficiently close to 1). On the other hand, the group $\Gamma=\mathbb{Z}_p^\times$ acts on each annuli $(r,1]$ isometrically by mapping $X\mapsto(1+X)^a-1:=\sum_{n\geq 1}\binom{a}{n}X^n$ (one can visualize this as some sort of rotation of the annuli $(r,1]$).These two actions then induce actions on $\mathcal{R}$: we have $\varphi(f(X))=f((1+X)^p-1)$ and $af(X)=f((1+X)^a-1)$. We define a (φ,Γ)-module to be a finite free $\mathrm{B}^\dagger_{\mathrm{rig}}$-module with commuting φ,Γ actions.

Now the ring $\mathrm{B}^\dagger_{\mathrm{rig}}$ has a very nice element $t=\log(1+X)$ on which Γ acts by $\gamma(t)=\gamma\cdot t$. For $n>>0$ we have localization maps $\iota_n:\mathrm{B}^{\dagger,r}_{\mathrm{rig}}\hookrightarrow\mathbb{Q}_p(\mu_{p^n})\left((t)\right)$, given by mapping $X\mapsto\zeta_{p^n}e^{\frac{t}{p^n}}-1$; essentially, these maps are taking the Taylor expansion at the point $\zeta_{p^n}-1$ with respect to the parameter t after sufficiently extending the domain of definition. For example, $\iota_n(t)=p^{-n}t$. Berger then defines a functor

$$\mathcal{M}:\{\text{Filtered }\varphi\text{-modules}\}\to\{(\varphi,\Gamma)\text{-modules}\},$$

$$\mathcal{M}\left(D\right)=\left\{y\in\mathrm{B}^\dagger_{\mathrm{rig}}[1/t]\otimes_{\mathbb{Q}_p}D:\iota_n(y)\in\mathrm{Fil}^0\left(\mathbb{Q}_p(\mu_{p^n})\left((t)\right)\otimes_{\mathbb{Q}_p}D\right)\right\}.$$

Moreover, given a crystalline representation V, we have $\mathcal{M}(\mathrm{D}_{\mathrm{cris}}(V))=\mathrm{D}(V)$, so $\mathrm{D}(\mathrm{Ku}_\alpha)=\mathcal{M}\left(\mathrm{D}_\alpha\right)$.

We have

$$\mathrm{Fil}^0\left(\mathbb{Q}_p(\mu_{p^n})\left((t)\right)\otimes_{\mathbb{Q}_p}D\right)=\mathbb{Q}_p(\mu_{p^n})(tx)+\mathbb{Q}_p(\mu_{p^n})\left(x\log_p\alpha+y\right).$$

Since

$$\iota_n\left(tx\right)=p^{-2n}tx\in\mathrm{Fil}^0\left(\mathbb{Q}_p(\mu_{p^n})\left((t)\right)\otimes_{\mathbb{Q}_p}D\right),$$

we have that $tx\in\mathcal{M}\left(D\right)$.

To find another independant element in $\mathcal{M}\left(D\right)$, we try to find one of the form $h\log_p\alpha\cdot x+y$, where $h\in\mathrm{B}^\dagger_{\mathrm{rig}}$.

Claim: there is an $h \in \mathrm{B}^{\dagger}_{\mathrm{rig}}$ with $h(\zeta_{p^n}-1)=p^n$ for $n>>0$.
Suppose the claim is true. For such an h, one has

$$\iota_n(h) \equiv_t h(\zeta_{p^n}-1)=p^n,$$

so that

$$\begin{aligned}\iota_n\left(h\log_p\alpha\cdot x+y\right) &\equiv_{\mod t} h(\zeta_{p^n}-1)\left(\log_p\alpha\right)p^{-n}x+y\\ &= x\log_p\alpha+y \in \mathrm{Fil}^0\left(\mathbb{Q}_p(\mu_{p^n})\left((t)\right)\otimes_{\mathbb{Q}_p} D\right).\end{aligned}$$

Thus, we have computed the (φ,Γ)-module $\mathrm{D}(\mathrm{Ku}_\alpha)=\mathcal{M}\left(\mathrm{D}_\alpha\right)$ contains

$$\mathcal{M}=\mathrm{B}^{\dagger}_{\mathrm{rig}}\left(tx\right)+\mathrm{B}^{\dagger}_{\mathrm{rig}}\left(h\log_p\alpha\cdot x+y\right).$$

Let us explain why this submodule is equal to $\mathcal{M}\left(\mathrm{D}_\alpha\right)$. Suppose $z\in\mathcal{M}\left(\mathrm{D}_\alpha\right)$; then for $n>>0$ we have

$$\iota_n(z)\in \mathrm{Fil}^0\left(\mathbb{Q}_p(\mu_{p^n})\left((t)\right)\otimes_{\mathbb{Q}_p} D\right)=\mathrm{Span}_{\mathbb{Q}_p(\mu_{p^n})}\left\{\iota_n(tx),\iota_n\left(h\log_p\alpha\cdot x+y\right)\right\}.$$

Because $\iota_n=\iota_{n-1}\circ\varphi$, this implies that we can find $h_1,h_2\in\mathrm{B}^{\dagger}_{\mathrm{rig}}$ such that for $n>>0$

$$\iota_n\left(z-h_1\left(tx\right)-h_2\left(\left(h\log_p\alpha\cdot x+y\right)\right)\right)\in t\mathrm{Fil}^0\left(\mathbb{Q}_p(\mu_{p^n})\left((t)\right)\otimes_{\mathbb{Q}_p} D\right).$$

Thus, we have shown that $\mathcal{M}\subset\mathcal{M}\left(D_\alpha\right)\subset\mathcal{M}+t\mathcal{M}\left(D_\alpha\right)$. Comparing $\det\mathcal{M}$ and $\det\mathcal{M}\left(D_\alpha\right)$, it follows that $\mathcal{M}=\mathcal{M}(D_\alpha)$.

Proof of the claim. For $n\geq 0$, let $\Phi_n\left(X\right)=(1+X)^{p^n}-1$ and $P_n(X)=\Phi_n(X)/p\Phi_{n-1}(X)$. Then $P_n(X)$ has roots $\zeta_{p^n}-1$ for ζ_{p^n} primitive, and $\log(1+X)=X\prod_{n\geq1}P_n(X)$. Consider

$$h_n(X):=\left(\frac{p}{\zeta_p-1}\right)p^{2n}\frac{\log(1+X)}{P_n(X)}.$$

By construction, $h_n(\zeta_{p^n}-1)=p^n$ while $h_n(\zeta_{p^m}-1)=0$ for $m\neq n$. Further, $h_n(X)$ converges on the entire disc so it lies in $\mathrm{B}^{\dagger}_{\mathrm{rig}}$. Finally, set $h(X)=\sum_{n\geq1}h_n(X)$. The sum converges because for $n>>0$ we have

$$|P_n(X)|=\left|\frac{1}{p}(1+(1+X)^{p^n}+...+(1+X)^{(p-1)p^n}\right|=1,$$

so

$$|h_n(X)|=|\log(1+X)|p^{-O(n)}.$$

5. Berger's theorem

Recall that if D is a (φ,Γ)-module over $\mathrm{B}^{\dagger}_{\mathrm{rig}}$ then the Γ-action can be differentiated to give a connection $\nabla_D=\frac{\log(\gamma)}{\log_p\chi_{\mathrm{cyc}}(\gamma)}$. On $f(X)\in\mathrm{B}^{\dagger}_{\mathrm{rig}}$ it acts by $\nabla(f)=t\cdot(1+X)\frac{df}{dX}$. A (φ,Γ)-module with a locally trivial connection is a (φ,Γ)-module D for which there is an r and n such that

$$\mathbb{Q}_p\left(\mu_{p^n}\right)\left((t)\right)\otimes^{\iota_n}_{\mathrm{B}^{\dagger,r}_{\mathrm{rig}}}D^{\dagger,r}=\mathbb{Q}_p\left(\mu_{p^n}\right)\left((t)\right)\otimes_{\mathbb{Q}_p\left(\mu_{p^n}\right)}\left(\mathbb{Q}_p\left(\mu_{p^n}\right)\left((t)\right)\otimes^{\iota_n}_{\mathrm{B}^{\dagger,r}_{\mathrm{rig}}}D^{\dagger,r}\right)^{\nabla\otimes\nabla_D=0}.$$

Any (φ,Γ)-module rising from the construction $D\mapsto\mathcal{M}(D)$ of the previous section is one which has a local trivial connection precisely because we have

$$\mathbb{Q}_p\left(\mu_{p^n}\right)\otimes_{\mathbb{Q}_p}D=\left(\mathbb{Q}_p\left(\mu_{p^n}\right)\left((t)\right)\otimes^{\iota_n}D\right)^{\nabla\otimes\nabla_D=0},$$

or in other words D itself is giving the solution to the differential equation $\nabla \otimes \nabla_D = 0$. In more generality than that of the previous section, the construction $D \mapsto \mathcal{M}(D)$ extends to an equivalence of categories

$$\left\{\text{Filtered } \left(\varphi, N, \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)\right)\right\} \xrightarrow{\sim} \{\ (\varphi, \Gamma)\text{-modules a with a locally trivial connection}\} .$$