Computing the \((\varphi, \Gamma)\)-module attached to a crystalline Kummer extension

GAL PORAT

Abstract. These are notes for a talk which computes the \((\varphi, \Gamma)\)-module attached to a crystalline Kummer extension.

1. Introduction

According to a theorem of Fontaine, Cherbonnier-Colmez and Kedlaya, there is a fully faithful functor
\[ D : \text{Representations } \rho : \text{Gal} \left( \overline{\mathbb{Q}}_p/\mathbb{Q}_p \right) \to \text{GL}_n \left( \mathbb{Q}_p \right) \hookrightarrow \text{\{(\varphi, \Gamma)\}-modules over the Robba ring} \].

However, given a \(p\)-adic representation \(V\), it is not clear at first how to compute the associated \((\varphi, \Gamma)\) module \(D(V)\). The point of this talk is to explain how to do this in a very simple case where \(V\) is a crystalline Kummer extension, following a method of Berger.

2. The crystalline Kummer extension

Recall that we have the \(\text{Gal} \left( \overline{\mathbb{Q}}_p/\mathbb{Q}_p \right)\)-equivariant Kummer sequence:
\[ 0 \to \mu_{p^n} \left( \overline{\mathbb{Q}}_p \right) \to \overline{\mathbb{Q}}_p^\times p^n \xrightarrow{\cdot p^n} \overline{\mathbb{Q}}_p^\times \to 0. \]

Taking Galois cohomology, this gives
\[ \mathbb{Q}_p^\times \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p^n \xrightarrow{\sim} H^1 \left( \text{Gal} \left( \overline{\mathbb{Q}}_p/\mathbb{Q}_p \right), \mu_{p^n} \right), \]
with the isomorphism being
\[ \alpha \otimes 1 \mapsto \xi_\alpha, \]
where \(\xi_\alpha\) is determined by the identity
\[ \zeta_{p^n}(g) = \frac{g(\alpha^{1/p^n})}{\alpha^{1/p^n}}, g \in \text{Gal} \left( \overline{\mathbb{Q}}_p/\mathbb{Q}_p \right). \]

Taking the limit and tensoring with \(\mathbb{Q}_p\), we get the Kummer isomorphism
\[ \text{Ku} : \mathbb{Q}_p^\times \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} H^1 \left( \text{Gal} \left( \overline{\mathbb{Q}}_p/\mathbb{Q}_p \right), \mathbb{Q}_p(1) \right). \]

Given an element \(x \in \mathbb{Q}_p\), write \(x^\flat\) for a compatible sequence of \(p^n\)'th roots of \(x\). For example, we shall write \(1^\flat = (1, \zeta_p, \zeta_p^2, ...)\) (nonstandard notation).

With this notation, the isomorphism \(\text{Ku}\) can be described concisely as mapping \(\alpha \otimes 1 \mapsto \xi_\alpha\) where \(1^\flat \zeta_\alpha(g) = \alpha^{g(\alpha)}\) for \(g \in \text{Gal} \left( \overline{\mathbb{Q}}_p/\mathbb{Q}_p \right)\). The crystalline Kummer extensions are these Kummer extensions which correspond to \(\alpha\) being an integral unit, i.e.
\[ \mathbb{Z}_p^\times \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \{\text{Crystalline Kummer extensions}\}. \]
In terms of representations, $\alpha \otimes 1$ corresponds to the representation

$$
g \in \text{Gal} \left( \overline{\mathbb{Q}}_p / \mathbb{Q}_p \right) \mapsto \begin{pmatrix}
\chi_{\text{cyc}}(g) & \xi_\alpha(g) \\
0 & 1
\end{pmatrix}.
$$

Let us write $\text{Ku}_\alpha$ for this representation. Since $\mathbb{Z}_p^\times \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = (1 + p\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ we will always assume $\alpha \in 1 + p\mathbb{Z}_p$.

3. Filtered $\varphi$-modules

A filtered $\varphi$-module is a $\mathbb{Q}_p$-vector space $D$, together with a linear isomorphism $\varphi : D \xrightarrow{\sim} D$ and a separated, exhaustive decreasing filtration

$$
\ldots \supset \text{Fil}^{-1} D \supset \text{Fil}^0 D \supset \text{Fil}^1 D \supset \ldots
$$

There is a functor

$$
D_{\text{cris}} : \{ \text{Representations } \rho : \text{Gal} \left( \overline{\mathbb{Q}}_p / \mathbb{Q}_p \right) \to \text{GL}_n \left( \mathbb{Q}_p \right) \} \to \{ \text{Filtered } \varphi\text{-modules} \}.
$$

When restricted to crystalline representations, such as the crystalline Kummer extensions, this functor is actually a fully faithful embedding.

The functor $D_{\text{cris}}$ is defined as follows. There is a $\mathbb{Q}_p$-algebra called $B_{\text{cris}}$; this ring has several structures: it has an action of $\text{Gal} \left( \overline{\mathbb{Q}}_p / \mathbb{Q}_p \right)$, an action of $\varphi$ and a filtration. We then set

$$
D_{\text{cris}} (V) := \left( B_{\text{cris}} \otimes_{\mathbb{Q}_p} V \right)^{\text{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)}.
$$

To keep this talk at a reasonable length, we won’t define $B_{\text{cris}}$ here. Instead, let us point out a property of $B_{\text{cris}}$ that is not difficult to verify from the definition. In fact this property gives quite a bit of motivation for the definition of $B_{\text{cris}}$. It is as follows: given an element of $x \in 1 + p\mathbb{Z}_p$, there is a multiplicative element $[x^3] \in B_{\text{cris}}$ so that $[x^3] [y^3] = [x^3 y^3]$ and an element $\log [x^3] \in B_{\text{cris}}$ satisfying the identities one expects $\log$ to satisfy.

For example, there is an element $t := \log [1^3]$, so that

$$
g(t) = g \left( \log [1^3] \right) = \log \left( (1^3)^{\chi_{\text{cyc}}(g)} \right) = \log \left( (1^3) \right)^{\chi_{\text{cyc}}(g)} = \chi_{\text{cyc}}(g) \log [1^3] = \chi_{\text{cyc}}(g) t.
$$

Now let us compute $D_{\text{cris}} (\text{Ku}_\alpha)$. By definition,

$$
D_{\text{cris}} (\text{Ku}_\alpha) = \left( B_{\text{cris}} \otimes_{\mathbb{Q}_p} \text{Ku}_\alpha \right)^{\text{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)}.
$$

By definition, we have a basis $e, f$ of the 2-dimensional representation $\text{Ku}_\alpha$ for which the action is given by

$$
g(e) = \chi_{\text{cyc}}(g) e, \quad g(f) = f + \xi_\alpha(g) e.
$$

We need to find a 2-dimensional $\mathbb{Q}_p$-subspace in $B_{\text{cris}} \otimes_{\mathbb{Q}_p} \text{Ku}_\alpha$ fixed by $\text{Gal} \left( \overline{\mathbb{Q}}_p / \mathbb{Q}_p \right)$. First, we notice that

$$
g(t^{-1} \otimes e) = \chi_{\text{cyc}}(g)^{-1} t^{-1} \otimes \chi_{\text{cyc}}(g) e = t^{-1} \otimes e,
$$

so $t^{-1} \otimes e$ is a first such vector. To find another fixed vector, it is natural guess there is one of the form $f + t^{-1} b \otimes e$ with $b \in B_{\text{cris}}$. Then for $f + t^{-1} b \otimes e$ to be fixed by $\text{Gal} \left( \overline{\mathbb{Q}}_p / \mathbb{Q}_p \right)$ we
need to have \( g(b) - b = -\xi_\alpha(g)t \). Where can we find such an element? The trick is to take log \([\cdot]\) of both sides of the identity \((1^p)^\xi_\alpha(g) = \frac{g(\alpha^t)}{\alpha^t} : \)

\[
g \left( \log \left[ \alpha^t \right] \right) = \log \left[ \alpha^t \right] + \xi_\alpha t,
\]

so that \( f + t^{-1}b \otimes e \) is fixed for \( b = -\log \left[ \alpha^t \right] \).

**Summary:** writing \( x = t^{-1} \otimes e \) and \( y = f - \log \left[ \alpha^t \right] t^{-1} \otimes e \) we have \( D_\alpha := D_{\text{cris}}(K_{\alpha}) = \mathbb{Q}_p(\pi) \oplus \mathbb{Q}_p(y), \) with \( \varphi(x) = p^{-r}x \) and \( \varphi(y) = y \). The filtration is determined by how many times elements are divisible by \( t \), and since \( \log \left[ \alpha^t \right] \equiv \log_p \alpha \mod t \), we have

\[
\text{Fil}^{-1}D_\alpha = \text{Fil}^{0}D_\alpha = \mathbb{Q}_p \left( x \log_p \alpha + y \right) \supset \text{Fil}^{1}D_\alpha = 0.
\]

4. \((\varphi, \Gamma)\)-modules

Recall the Robba ring \( B^\dagger_{\text{rig}} \) is the ring of power series

\[
B^\dagger_{\text{rig}} = \left\{ f(X) = \sum_{n \in \mathbb{Z}} a_n X^n : a_n \in \mathbb{Q}_p, f(X) \text{ converges in some annulus } r < |X| \leq 1 \right\}.
\]

There are two natural operations we can perform on an annuli \((r, 1]\). There is the operation mapping \( \varphi : X \mapsto (1 + X)^p - 1 \), which inflates the annuli \((r, 1] \) to the larger annuli \((r^p, 1] \) (at least, if \( r \) is sufficiently close to 1). On the other hand, the group \( \Gamma = \mathbb{Z}_p^* \) acts on each annuli \((r, 1]\) isometrically by mapping \( X \mapsto (1 + X)^a - 1 := \sum_{n \geq 1} \binom{a}{n}X^n \) (one can visualize this as some sort of rotation of the annuli \((r, 1]\)). These two actions then induce actions on \( \mathcal{R} : \) we have \( \varphi(f(x)) = f((1 + X)^p - 1) \) and \( af(x) = f((1 + X)^a - 1) \). We define a \((\varphi, \Gamma)\)-module to be a finite free \( B^\dagger_{\text{rig}} \)-module with commuting \( \varphi, \Gamma \) actions.

Now the ring \( B^\dagger_{\text{rig}} \) has a very nice element \( t = \log(1 + X) \) on which \( \Gamma \) acts by \( \gamma(t) = \gamma \cdot t \). For \( n >> 0 \) we have localization maps \( \iota_n : B^\dagger_{\text{rig}} \rightarrow \mathbb{Q}_p(\mu_{p^n})((t)) \), given by mapping \( X \mapsto \zeta_{p^n} e^{\frac{r}{p^n} - 1} \); essentially, these maps are taking the Taylor expansion at the point \( \zeta_{p^n} - 1 \) with respect to the parameter \( t \) after sufficiently extending the domain of definition. For example, \( \iota_n(t) = p^{-n}t \). Berger then defines a functor

\[
\mathcal{M} : \{\text{Filtered } \varphi\text{-modules}\} \rightarrow \{((\varphi, \Gamma)\text{-modules}\},
\]

\[
\mathcal{M}(D) = \left\{ y \in B^\dagger_{\text{rig}}[1/t] \otimes_{\mathbb{Q}_p} D : \iota_n(y) \in \text{Fil}^0 \left( \mathbb{Q}_p(\mu_{p^n})((t)) \otimes_{\mathbb{Q}_p} D \right) \right\}.
\]

Moreover, given a crystalline representation \( V \), we have \( \mathcal{M}(D_{\text{cris}}(V)) = D(V) \), so \( D(K_{\alpha}) = \mathcal{M}(D_{\alpha}) \).

We have

\[
\text{Fil}^0 \left( \mathbb{Q}_p(\mu_{p^n})((t)) \otimes_{\mathbb{Q}_p} D \right) = \mathbb{Q}_p(\mu_{p^n})(tx) + \mathbb{Q}_p(\mu_{p^n}) \left( x \log_p \alpha + y \right).
\]

Since

\[
\iota_n \left( tx \right) = p^{-2n}tx \in \text{Fil}^0 \left( \mathbb{Q}_p(\mu_{p^n})((t)) \otimes_{\mathbb{Q}_p} D \right),
\]

we have that \( tx \in \mathcal{M}(D) \).

To find another independent element in \( \mathcal{M}(D) \), we try to find one of the form \( h \log_p \alpha \cdot x + y \), where \( h \in B^\dagger_{\text{rig}} \).
Claim: there is an \( h \in B^1_{\text{rig}} \) with \( h(\zeta_{p^n} - 1) = p^n \) for \( n \gg 0 \).

Suppose the claim is true. For such an \( h \), one has

\[
t_n(h) \equiv h(\zeta_{p^n} - 1) = p^n,
\]

so that

\[
t_n \left( h \log_p \alpha \cdot x + y \right) \equiv \text{mod } t \left( h(\zeta_{p^n} - 1) \left( \log_p \alpha \right) p^{-n} x + y \right) = x \log_p \alpha + y \in \text{Fil}^0 \left( Q_p \left( \mu_{p^n} \right) \left( (t) \right) \otimes Q_p \right) D .
\]

Thus, we have computed the \((\varphi, \Gamma)\)-module \( D(Ku_\alpha) = \mathcal{M}(D_\alpha) \) contains

\[
\mathcal{M} = B^1_{\text{rig}}(tx) + B^1_{\text{rig}} \left( h \log_p \alpha \cdot x + y \right).
\]

Let us explain why this submodule is equal to \( \mathcal{M}(D_\alpha) \). Suppose \( z \in \mathcal{M}(D_\alpha) \); then for \( n > 0 \) we have

\[
t_n(z) \in \text{Fil}^0 \left( Q_p \left( \mu_{p^n} \right) \left( (t) \right) \otimes Q_p \right) D = \text{Span}_{Q_p(\mu_{p^n})} \left\{ t_n(tx), t_n \left( h \log_p \alpha \cdot x + y \right) \right\}.
\]

Because \( t_n = t_{n-1} \circ \varphi \), this implies that we can find \( h_1, h_2 \in B^1_{\text{rig}} \) such that for \( n > 0 \)

\[
t_n \left( z - h_1(tx) - h_2 \left( ( (h \log_p \alpha \cdot x + y) ) \right) \right) \in t\text{Fil}^0 \left( Q_p \left( \mu_{p^n} \right) \left( (t) \right) \otimes Q_p \right) D .
\]

Thus, we have shown that \( \mathcal{M} \subseteq \mathcal{M}(D_\alpha) \subseteq \mathcal{M} + t\mathcal{M}(D_\alpha) \). Comparing \( \det \mathcal{M} \) and \( \det \mathcal{M}(D_\alpha) \), it follows that \( \mathcal{M} = \mathcal{M}(D_\alpha) \).

Proof of the claim. For \( n \geq 0 \), let \( \Phi_n(X) = (1 + X)^{p^n} - 1 \) and \( P_n(X) = \Phi_n(X)/p\Phi_{n-1}(X) \). Then \( P_n(X) \) has roots \( \zeta_{p^n} - 1 \) for \( \zeta_{p^n} \) primitive, and \( \log(1 + X) = X \prod_{n \geq 1} P_n(X) \).

Consider

\[
h_n(X) := \left( \frac{p}{\zeta_{p^n} - 1} \right) p^{2n} \log(1 + X) / P_n(X).
\]

By construction, \( h_n(\zeta_{p^n} - 1) = p^n \) while \( h_n(\zeta_{p^m} - 1) = 0 \) for \( m \neq n \). Further, \( h_n(X) \) converges on the entire disc so it lies in \( B^1_{\text{rig}} \). Finally, set \( h(X) = \sum_{n \geq 1} h_n(X) \). The sum converges because for \( n > 0 \) we have

\[
|P_n(X)| = \left| \frac{1}{p} \frac{1}{(1 + (1 + X)^{p^n} + \cdots + (1 + X)^{(p-1)p^n})} \right| = 1,
\]

so

\[
|h_n(X)| = |\log(1 + X)| p^{-O(n)}.
\]

5. Berger’s theorem

Recall that if \( D \) is a \((\varphi, \Gamma)\)-module over \( B^1_{\text{rig}} \), then the \( \Gamma \)-action can be differentiated to give a connection \( \nabla_D = \frac{\log(\gamma)}{\log_p \chi_{\text{cyc}}(\gamma)} \). On \( f(X) \in B^1_{\text{rig}} \) it acts by \( \nabla(f) = t \cdot (1 + X)^{df/dX} \). A \((\varphi, \Gamma)\)-module with a locally trivial connection is a \((\varphi, \Gamma)\)-module \( D \) for which there is an \( r \) and \( n \) such that

\[
Q_p \left( \mu_{p^n} \right) \left( (t) \right) \otimes_{B^1_{\text{rig}}} D^{t,r} = Q_p \left( \mu_{p^n} \right) \left( (t) \right) \otimes_{Q_p(\mu_{p^n})} \left( Q_p \left( \mu_{p^n} \right) \left( (t) \right) \otimes_{B^1_{\text{rig}}} D^{t,r} \right)^{\nabla \otimes \nabla_D = 0}.
\]

Any \((\varphi, \Gamma)\)-module rising from the construction \( D \mapsto \mathcal{M}(D) \) of the previous section is one which has a local trivial connection precisely because we have

\[
Q_p \left( \mu_{p^n} \right) \otimes_{Q_p} D = \left( Q_p \left( \mu_{p^n} \right) \left( (t) \right) \otimes_{B^1_{\text{rig}}} D \right)^{\nabla \otimes \nabla_D = 0},
\]
or in other words $D$ itself is giving the solution to the differential equation $\nabla \otimes \nabla_D = 0$. In more generality than that of the previous section, the construction $D \mapsto \mathcal{M}(D)$ extends to an equivalence of categories

$$\{\text{Filtered } (\varphi, N, \text{Gal } (\overline{\mathbb{Q}}_p/\mathbb{Q}_p))\} \xrightarrow{\sim} \{ (\varphi, \Gamma)\text{-modules a with a locally trivial connection} \}.$$