COMPLETED COHOMOLOGY: BASIC EXAMPLES AND COMPUTATIONS

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Abstract. This note contains some basic examples and computations in completed cohomology.

1. The set up

Let $\mathbb{G}$ be a reductive group over $\mathbb{Q}$, let K_∞ be a maximal compact open of $G_\infty = \mathbb{G}(\mathbb{R})$, A_∞ the $\mathbb{R}$-points of a maximal $\mathbb{Q}$-split torus of the center of $\mathbb{G}$ (this has $A_\infty^\circ = 1$ if $\mathbb{G}$ is semisimple), let $K^p \subset \mathbb{G}(\mathbb{A}^p)$ a fixed compact open (tame level) and let $K_{r,p}$ for $r \geq 1$ be a system of compact mormal open neighborhoods of the identity in $\mathbb{G}(\mathbb{Q}_p)$. One forms the locally symmetric spaces

$$Y_r = \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/K_\infty^\circ A_\infty^\circ K^p K_{r,p}.$$

(Note that if we let $X^\circ = G_\infty^\circ/K_\infty^\circ A_\infty^\circ$, each Y_r is a finite disjoint union of quotients of X by congruences subgroups. In particular $\dim Y_r = \dim X = d$ (independently of r) as a real manifold).

If $K_p = K_{0,p}$ is sufficiently small, we get a tower $... \to Y_1 \to Y_0$ where all the maps $Y_m \to Y_n$ are Galois coverings with group $K_{n,p}/K_{m,p}$.

The completed cohomology of $\mathbb{G}$ of tame level K^p is then defined to be

$$\widetilde{\mathrm{H}}^i\left(\mathbb{G}, K^p\right) = \lim_{s\to\infty} \operatorname*{colim}_{r\to\infty} \mathrm{H}^i\left(Y_r, \mathbb{Z}/p^s\right).$$

We also have completed homology

$$\widetilde{\mathrm{H}}_i\left(\mathbb{G}, K^p\right) = \lim_{r\to\infty} \mathrm{H}_i\left(Y_r, \mathbb{Z}_p\right).$$

One sometimes considers the variant

$$\widetilde{\mathrm{H}}^i\left(\mathbb{G}\right) = \mathrm{colim}_{K^p} \widetilde{\mathrm{H}}^i\left(\mathbb{G}, K^p\right).$$

2. What kind of an object is completed cohomology?

The object $\widetilde{\mathrm{H}}^i(\mathbb{G})$ is a obviously a $\mathbb{Z}_p$-module. It is endowed with actions the big p-adic Hecke algebra $\mathbb{T}$ and the adelic group $\mathbb{G}(\mathbb{A})$ (at infinity, this action factors through the action of the connected component of $G_\infty/A_\infty^\circ K_\infty^\circ$). If $\mathbb{G}$ is such that the $Y(K_f)$ admit the structure of a shimura variety (but not otherwise) then it also has an action of a Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$, where E is something like the reflexe field. It is a unit ball in a very large Banach space $\widetilde{\mathrm{H}}^i(\mathbb{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, which is reminiscient of the Banach spaces occuring in the cohomology of perfectoid spaces (made precise by Scholze). Its rationalization $\widetilde{\mathrm{H}}^i(\mathbb{G})_{\mathbb{Q}_p} = \widetilde{\mathrm{H}}^i(\mathbb{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

is a p-adic analogue of $L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}))$ which sees cuspidal algebraic automorphic forms as well as other non algebraic objects; indeed, it is complete, and the Hecke eigenspaces of $\widetilde{\mathrm{H}}^i(\mathbb{G})_{\mathbb{Q}_p}$ know about p-adic systems of Hecke eigenvalues, similalrly to what happens to $L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}))$. Both are global objects with local coefficients. Moreover, both have a dense set of interest: in $L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}))$ these are the automorphic forms $\mathcal{A}(\mathbb{G})$, for which the action at ∞ is subsituted for the action of a $(\mathfrak{g}, K)$-module; in $\widetilde{\mathrm{H}}^i(\mathbb{G})_{\mathbb{Q}_p}$ it's the locally analytic vectors $\widetilde{\mathrm{H}}^i(\mathbb{G})^{\mathrm{la}}_{\mathbb{Q}_p}$, which now have an action of $\mathfrak{gl}_2(\mathbb{Q}_p)$. The subset $\widetilde{\mathrm{H}}^i(\mathbb{G})^{\mathrm{la}}_{\mathbb{Q}_p}$ is a direct limit of (much) smaller Banach spaces. The locally algebraic objects are supposed to capture information about the algebraic systems of Hecke eigenvalues, or what is conjecturally the same, geometric Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\overline{\mathbb{Q}}_p$ coefficients (again, similar to the case of $L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}))$, where algebraicity should have something to do with the parameters of the $(\mathfrak{g}, K)$-action to be integers).

3. Some general properties

3.1. **How are the actions defined?** The Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ (when it exists) is induced from the structure of etale cohomology on each $\mathrm{H}^i(Y_r, \mathbb{Z}/p^s)$.

The adelic group acts as follows. At infinity, G_∞ acts through $G_\infty \twoheadrightarrow G_\infty/G_\infty^\circ = \pi_0(G_\infty)$, i.e. the action which swaps connected components of $X = G_\infty/K_\infty^\circ A_\infty^\circ$, which then induces an action on each Y_r. (For example, for $\mathbb{G} = \mathrm{GL}_2$ we have $X = \mathbb{H}^\pm$ and the action of $\pi_0 \cong \{\pm 1\}$ is the conjugation which swaps between the components).

At each $g \in \mathbb{G}(\mathbb{A}_f)$, one has a map action $Y(K_f) \to Y(g^{-1}K_f g)$ by mapping $\mathbb{G}(\mathbb{Q})\gamma K_\infty^\circ A_\infty^\circ K_f$ to $\mathbb{G}(\mathbb{Q})(\gamma g) K_\infty^\circ A_\infty^\circ (g^{-1}K_f g)$. If K_f is normal (which we will always assume), then this is a map from $Y(K_f) \to Y(K_f)$, so it induces the map on cohomology. Since we are keeping K^p fixed, the action of $\mathbb{G}(\mathbb{A}_f^p)$ on $\widetilde{\mathrm{H}}^i(\mathbb{G})$ is smooth, i.e. each element in cohomology is fixed by a compact open. Indeed if if $g \in K^p$ then the map $Y(K_f) \to Y(g^{-1}K_f g)$ is not doing anything, so elements of $\widetilde{\mathrm{H}}^i(\mathbb{G}, K^p)$ are fixed by K^p (and in fact $\widetilde{\mathrm{H}}^i(\mathbb{G})^{K^p} = \widetilde{\mathrm{H}}^i(\mathbb{G}, K^p)$). The action of $\mathbb{G}(\mathbb{Q}_p)$ is not smooth, but rather has some interesting locally analytic vectors which are not smooth.

As for the Hecke algebra, I think this goes as follows. Recall local p-adic Hecke algebras are just locally constant functions on $\mathbb{Z}_p[K_l\backslash\mathbb{G}(\mathbb{Q}_l)/K_l]$. If K_l is sufficiently generic (hyperspecial and $\mathbb{G}$ is unramified at l) then the Satake isomorphism says this is some symmetric polynomial algebra. Now the p-adic spherical Hecke algebra $\mathcal{H}^{\mathrm{sph}}(K^p)$ is the restricted tensor product of the local Hecke algebras at these sufficiently generic places. We stash all the rest of the places away from p rest into a tensor product $\mathcal{H}^{\mathrm{ram}}(K^p)$ which is usually not commutative, and set $\mathcal{H} = \mathcal{H}^{\mathrm{ram}}(K^p) \otimes_{\mathbb{Z}_p} \mathcal{H}^{\mathrm{sph}}(K^p)$. This acts on cohomology as follows. Basically every element here is a finite sum of elements $1_{K^p g K^p}$ for $g \in \mathbb{G}(\mathbb{A}_f^p)$. One can always write $K^p g K^p = \amalg g_i K^p$ for a finite disjoint union and $g_i \in \mathbb{G}(\mathbb{A}_f^p)$. Then each such element acts on $\mathrm{H}^i(Y(K_f))$ (with any coefficients) by mapping $x \mapsto \sum_i g_i^* x$ (the action of g_i^* induced from that on $Y(K_f)$ explained above). This action on $\widetilde{\mathrm{H}}^i(\mathbb{G}, K^p)$ factors through the smaller Hecke algebra $\mathbb{T}$ which is just what you get when you mod out by the relations you get when acting on all finite cohomology $\mathrm{H}^i(Y(K^p K_{p,r}))$, because of the spectral sequence explained in 3.4 below.

3.2. **Duality.** Up to p-power torsion, the groups $\widetilde{\mathrm{H}}^i(\mathbb{G}, K^p)$ and $\widetilde{\mathrm{H}}_i(\mathbb{G}, K^p)$ are duals; there are short exact sequences

$$0 \to \mathrm{Hom}_{\mathrm{cont}}\left(\widetilde{\mathrm{H}}_{i-1}(\mathbb{G}, K^p), \mathbb{Q}_p/\mathbb{Z}_p\right) \to \widetilde{\mathrm{H}}^i(\mathbb{G}, K^p) \to \mathrm{Hom}_{\mathrm{cont}}\left(\widetilde{\mathrm{H}}_i(\mathbb{G}, K^p), \mathbb{Z}_p\right) \to 0$$

and

$$0 \to \mathrm{Hom}_{\mathrm{cont}}\left(\widetilde{\mathrm{H}}^{i+1}(\mathbb{G}, K^p), \mathbb{Q}_p/\mathbb{Z}_p\right) \to \widetilde{\mathrm{H}}_i(\mathbb{G}, K^p) \to \mathrm{Hom}_{\mathrm{cont}}\left(\widetilde{\mathrm{H}}^i(\mathbb{G}, K^p), \mathbb{Z}_p\right) \to 0$$

. There are also Poincare duality spectral sequences which are best expressed in terms of homology (here $\widetilde{\mathrm{H}}_i^{\mathrm{BM}}$ means Borel-Moore completed homology, or compactly supported completed homology)

$$E_2^{i,j} := \mathrm{Ext}^i(\widetilde{\mathrm{H}}_j, \mathbb{Z}_p[[K_p]]) \Rightarrow \widetilde{\mathrm{H}}_{d-i-j}^{\mathrm{BM}}$$

and

$$E_2^{i,j} := \mathrm{Ext}^i(\widetilde{\mathrm{H}}_j^{\mathrm{BM}}, \mathbb{Z}_p[[K_p]]) \Rightarrow \widetilde{\mathrm{H}}_{d-i-j},$$

where d is the dimension of the symmetric space $\mathbb{G}_\infty/A_\infty^\circ K_\infty^\circ$.

3.3. **The basic structure.** The completed cohomology objects $\widetilde{\mathrm{H}}^i(\mathbb{G}, K^p)$ are admissible representations of $\mathbb{G}(\mathbb{Q}_p)$; dually, $\widetilde{\mathrm{H}}_i(\mathbb{G}, K^p)$ are finitely generated $\mathbb{Z}_p[[K_p]]$-modules. Once we invert p, we get something which fits better into the paradigm of locally analytic representation theory. The groups $\widetilde{\mathrm{H}}^i(\mathbb{G}, K^p)_{\mathbb{Q}_p}$ and $\widetilde{\mathrm{H}}_i(\mathbb{G}, K^p)_{\mathbb{Q}_p}$ are now duals; the group $\widetilde{\mathrm{H}}^i(\mathbb{G}, K^p)_{\mathbb{Q}_p}$ is a $\mathcal{C}(K_p, \mathbb{Q}_p)$-module which injects into a finite sum $\mathcal{C}(K_p, \mathbb{Q}_p)^{\oplus d}$ while $\widetilde{\mathrm{H}}_i(\mathbb{G}, K^p)_{\mathbb{Q}_p}$ is a finitely generated $\mathbb{Z}_p[[K_p]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module, and they are linked via the Schneider-Teitelbaum formalism for continuous representations.

3.4. **The relation to classical cohomology.** For each $r \geq 0$ there are spectral sequences

$$E_2^{i,j} := \mathrm{H}^i(K_{p,r}, \widetilde{\mathrm{H}}^j) \Rightarrow \mathrm{H}^{i+j}(Y_r, \mathbb{Z}_p).$$

. These sequences are Hecke equivariant, which leads eventually to the realization that all the classical cohomological systems of Hecke eigenvalues can be seen in completed cohomology, and that if these sequence degenerate sufficiently then in fact completed cohomology will be the closure of the classical cohomlogical systems.

4. Special cases

4.1. **Top degree completed cohomology and completed homology.** Let's call the symmetric space $X^\circ = G_\infty^\circ/A_\infty^\circ K_\infty^\circ$, and suppose $d > 0$. The set Y_r is a finite disjoint union of spaces of the form $\Gamma_r\backslash X^\circ$. If these spaces are not compact then each $\mathrm{H}_d(Y_r, \mathbb{Z}_p) = 0$ by general theorems. So in this case $\widetilde{\mathrm{H}}_d = 0$. We also have $\mathrm{H}^d(Y_r, \mathbb{Z}) = 0$ and it follows from the universal coefficient theorem for cohomology that $\widetilde{\mathrm{H}}^d = 0$. If they are compact, then we can use the fundamental class; it is basically multiplied by $\#K_{r,p}/K_{r+1,p}$ when passing from Y_{r+1} to Y_r or vice versa; so we still get vanishing $\widetilde{\mathrm{H}}^d = \widetilde{\mathrm{H}}_d = 0$.

Summarizing: one always has $\widetilde{\mathrm{H}}^d = \widetilde{\mathrm{H}}_d = 0$, if $d > 1$.

4.2. **Zero degree completed cohomology and homology.** We first compute the completed homology in degree 0 in several steps.

Lemma 4.1. *We have*

$$\widetilde{\mathrm{H}}_0(\mathbb{G}, K^p) = \mathbb{Z}_p\left[\left[\pi_0(Y(K^p))\right]\right] := \lim_{\infty \leftarrow r} \mathbb{Z}_p\left[\pi_0(Y(K^p K_{p,r}))\right].$$

Proof. This follows immediately from the definition, since $\widetilde{\mathrm{H}}_0(Y, \mathbb{Z}_p) = \mathbb{Z}_p\left[\pi_0(Y(K^p K_p))\right]$. □

Now let $\mathbb{G}(\mathbb{Q})^\circ = \mathbb{G}(\mathbb{R})^\circ \cap \mathbb{G}(\mathbb{Q})$.

Proposition 4.1. *We have $\pi_0(Y(K_f)) = \mathbb{G}(\mathbb{Q})^\circ \backslash \mathbb{G}(\mathbb{A}^\infty)/K_f$, an equality of pointed sets.*

Proof. Let $X = \mathbb{G}(\mathbb{R})/A_\infty^\circ K_\infty^\circ$. Then

$$Y(K_f) = \mathbb{G}(\mathbb{Q}) \backslash (X \times \mathbb{G}(\mathbb{A}^\infty))/K_f \cong \mathbb{G}(\mathbb{Q})^\circ \backslash (X^\circ \times \mathbb{G}(\mathbb{A}^\infty))/K_f,$$

the isomorphism holding because $\mathbb{G}(\mathbb{Q})$ is dense in $G(\mathbb{R})$. Now by a theorem of Borel, $\mathbb{G}(\mathbb{Q})^\circ \backslash \mathbb{G}(\mathbb{A}^\infty)/K_f$ is finite, so if we write γ_i for the representatives, we see that

$$Y(K_f) = \amalg_i \mathbb{G}(\mathbb{Q})^\circ \cap \gamma_i K_f \gamma_i^{-1} \backslash X^\circ,$$

and each $\mathbb{G}(\mathbb{Q})^\circ \cap \gamma_i K_f \gamma_i^{-1} \backslash X^\circ$ is connected, being a quotient of X°. So the connected componets are indeed indexed by $\mathbb{G}(\mathbb{Q})^\circ \backslash \mathbb{G}(\mathbb{A}^\infty)/K_f$, where the trivial coset corresponding to the point of $\pi_0(Y(K_f))$. □

Proposition 4.2. *If K_f is sufficiently small and $K_f' \subset K_f$, then*

$$\pi_0(Y(K_f')) = \pi_0(Y(K_f)) \times \left(\mathbb{G}(\mathbb{Q})^\circ \cap K_f \backslash K_f/K_f'\right).$$

Proof. If K_f is sufficiently small then $Y(K_f') \to Y(K_f)$ is a Galois covering with Galois group K_f/K_f'; in particular, every connected component of $Y(K_f)$ has an equal amount of connected components of $Y(K_f')$ being sent to it. To find out what it is, it's enough to find the fiber over the connected component of the trivial coset of $\pi_0(Y(K_f)) = \mathbb{G}(\mathbb{Q})^\circ \backslash \mathbb{G}(\mathbb{A}^\infty)/K_f$. Clearly, this is just the image of K_f in $\mathbb{G}(\mathbb{Q})^\circ \backslash \mathbb{G}(\mathbb{A}^\infty)/K_f'$, which is $\mathbb{G}(\mathbb{Q})^\circ \cap K_f \backslash K_f/K_f'$. □

Theorem 4.1. *For K_p fixed and sufficiently small, there is some finite group Δ such that*

$$\widetilde{\mathrm{H}}_0(\mathbb{G}, K^p) = \mathbb{Z}_p\left[\left[\Delta \times \left(\overline{\mathbb{G}(\mathbb{Q})^\circ \cap K_f} \backslash K_p\right)\right]\right] \cong \mathbb{Z}_p\left[\left[\left(\overline{\mathbb{G}(\mathbb{Q})^\circ \cap K_f} \backslash K_p\right)\right]\right]^\Delta,$$

where $\overline{\mathbb{G}(\mathbb{Q})^\circ \cap K_f}$ is defined as

$$\overline{\mathbb{G}(\mathbb{Q})^\circ \cap K_f} = \lim_{\infty \leftarrow r} \mathbb{G}(\mathbb{Q})^\circ \cap K_f/\mathbb{G}(\mathbb{Q})^\circ \cap K^p K_{p,r}.$$

Proof. Set $\Delta = \pi_0(Y(K_p K^p))$. According to Proposition 2.2 and Lemma 2.1, we have

$$\widetilde{\mathrm{H}}_0(\mathbb{G}, K^p) = \lim_{\infty \leftarrow r} \mathbb{Z}_p\left[\pi_0(Y(K^p K_{p,r}))\right] = \lim_{\infty \leftarrow r} \mathbb{Z}_p\left[\Delta \times (\mathbb{G}(\mathbb{Q})^\circ \cap K_f \backslash K_p/K_{p,r})\right].$$

Now, we have the exact sequence

$$1 \to \mathbb{G}(\mathbb{Q})^\circ \cap K^p K_p/\mathbb{G}(\mathbb{Q})^\circ \cap K^p K_{p,r} \to K_p/K_{p,r} \to \mathbb{G}(\mathbb{Q})^\circ \cap K^p K_p \backslash K_p/K_{p,r} \to 1.$$

Taking the inverse limit (and noting the Mittag-Leffler condition is satisfied), we see that the completion of $\mathbb{G}(\mathbb{Q})^\circ \cap K^p K_p \backslash K_p/K_{p,r}$ is $\overline{\mathbb{G}(\mathbb{Q})^\circ \cap K^p K_p} \backslash K_p$, as required. □

Now use duality to conclude that $\widetilde{\mathrm{H}}^0(\mathbb{G}, K^p) = \mathcal{C}\left(\Delta \times \left(\overline{\mathbb{G}(\mathbb{Q})^\circ \cap K_f} \backslash K_p\right), \mathbb{Z}_p\right)$.

Summarizing: $\widetilde{\mathrm{H}}_0(\mathbb{G}, K^p) = \mathbb{Z}_p\left[\left[\Delta \times \left(\overline{\mathbb{G}(\mathbb{Q})^\circ \cap K_f} \backslash K_p\right)\right]\right]$ and $\widetilde{\mathrm{H}}^0(\mathbb{G}, K^p) = \mathcal{C}\left(\Delta \times \left(\overline{\mathbb{G}(\mathbb{Q})^\circ \cap K_f} \backslash K_p\right)\right.$ for $\Delta = \pi_0(Y(K_pK^p))$, if K_p is sufficiently small.

5. The l_0 and q_0

One defines

$$l_0 = \mathrm{rank}\mathbb{G}_\infty - \mathrm{rank}A_\infty K_\infty$$

(when $\mathbb{G}$ is semisimple, this is the defect from having discrete series, i.e. a compact torus of full rank), and

$$q_0 = (\dim \mathbb{G}_\infty - \dim A_\infty K_\infty - l_0)/2.$$

Note that $\dim \mathbb{G}_\infty - \dim A_\infty K_\infty = \dim Y_r$.

Let us recall what the rank of a real Lie group. By definition, it is the dimension of a Cartan subalgebra of its Lie algebra. A Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra $\mathfrak{h}$that is self normalizing, meaning that if $[X, \mathfrak{h}] \subset \mathfrak{h}$ then $X \in \mathfrak{h}$. (In the semisimple case this is basically a maximal abelian subalgebra).

Calegari and Emerton have conjectures regarding to these quantities. Namely, define the codimension of a $\mathbb{Z}_p[[K_p]]$-module M to be the smallest $i \geq 0$ for which $\mathrm{Ext}^i(M, \mathbb{Z}_p[[K_p]]) \neq 0$. Then they conjecture that

$$\mathrm{codim}\widetilde{\mathrm{H}}_{q_0} = l_0,$$

that $\widetilde{\mathrm{H}}_i$ vanishes for $i > q_0$ and that $\widetilde{\mathrm{H}}_i$ has codimension greater than $l_0 + q_0 - i$ if $i < q_0$.

The conjecture that $\mathrm{codim}\widetilde{\mathrm{H}}_{q_0} = l_0$ is related to the expected Krull dimension of the Hecke algebra $\mathbb{T}$ being equal to $1 + \dim \mathbb{B} - l_0$, where $\mathbb{B}$ is a Borel subgroup of $\mathbb{G}$. One conjectures in fact that $\widetilde{\mathrm{H}}^{q_0}$ is something like a faithful module of $\mathbb{T}$.

6. Some examples

Here are some examples of things we have done so far.

6.1. $\mathbb{G} = \mathrm{GL}_1$. In this case X° is just a point so $d = 0$. Thus the completed homology and cohomology vanish above degree 0. In degree 0, we can compute what happens by our previous analysis. Let $K^p = K(N)$ for some N, i.e. the kernel of $\widehat{\mathbb{Z}}^{p\times} \to (\mathbb{Z}/N\mathbb{Z})^\times$ for some N coprime to p. Then if $K_{p,r} = 1 + p^{r+1}\mathbb{Z}_p$, we see that

$$\Delta = \pi_0(Y(K_pK^p)) = Y(K_pK^p) = (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times.$$

On the other hand we have (need to massage this a bit for $p = 2$)

$$\overline{\mathbb{G}(\mathbb{Q})^\circ \cap K_f} \backslash K_p = \overline{\{\pm 1\} \cap (1 + p\mathbb{Z}_p)} \backslash (1 + p\mathbb{Z}_p) = (1 + p\mathbb{Z}_p).$$

So we get that by section 4

$$\widetilde{\mathrm{H}}^0(\mathbb{G}, K^p) = \mathcal{C}\left(\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{Z}_p\right)$$

and

$$\widetilde{\mathrm{H}}_0(\mathbb{G}, K^p) = \mathbb{Z}_p\left[\left[\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times\right]\right].$$

It is easy to calculate that in this case $l_0 = q_0 = 0$, so everything fits in with the conjecture of Calegari and Emerton.

What about the structure? This should have actions of $\mathbb{A}^\times \times \mathbb{T}$ (this is not the case where we have an action of a subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$). The action of $\mathbb{A}^\times$ is just the right regular action on $\mathcal{C}\left(\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{Z}_p\right)$ by precomposing with the Grossencharacter

$$\mathbb{A}^\times \cong \mathbb{Q}^\times \times \mathbb{R}_{>0}^\times \times \widehat{\mathbb{Z}}^\times \twoheadrightarrow \widehat{\mathbb{Z}}^\times \twoheadrightarrow \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times .$$

The big Hecke algebra is just $\mathbb{T} = \mathbb{Z}_p\left[\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times\right]$, because $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ is the Galois group of the maximal abelian extension of level K^p (by the Kronecker Weber Theorem). It also acts via the right regular action (or maybe the left regular action, there might be a difference in the signs here). How is this seen from our description above? Well first we have the spherical Hecke algebra. At each $l \nmid Np$ we have the locally constant functions on $\mathbb{Z}_l^\times\backslash\mathbb{Q}_l^\times/\mathbb{Z}_l^\times \cong l^{\mathbb{Z}}$, so $\mathcal{H}^{\mathrm{sph}} = \mathbb{Z}_p\left[e_l\right]_{l\nmid Np}$, with e_l being the indicator function for the double coset $\mathbb{Z}_l^\times \cdot l \cdot \mathbb{Z}_l^\times$. The ramified part is simply $\mathbb{Z}_p\left[e_l\right]_{l|N}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right]$. Altogether we get that $\mathcal{H} = \mathbb{Z}_p\left[e_l\right]_{l\neq p}\left[(\mathbb{Z}/N\mathbb{Z})^\times\right]$. Now $Y(K_{p,r}K^p) = (\mathbb{Z}/p^{r+1}\mathbb{Z})^\times \times (\mathbb{Z}/N)^\times$, and the action of e_lK^p on this is thus given by multiplication with l on both coordinates. Thus e_l acts on $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ by multiplication with l, and hence acts on $\widetilde{\mathrm{H}}^0\left(\mathbb{G}, K^p\right) = \mathcal{C}\left(\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{Z}_p\right)$ by precomposing with l. This shows the action of $\mathbb{Z}_p\left[\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times\right]$ is also given by the right regular action.

In a sense this example is pretty much saying that $\widetilde{\mathrm{H}}_0$ *is* $\mathbb{T}$, and we really see it's a faithful module for it.

If we take the limit over all K^p, like one sometimes does, we see that

$$\widetilde{\mathrm{H}}^0\left(\mathrm{GL}_1\right) \cong \mathcal{C}\left(\mathbb{Z}_p^\times, \mathbb{Z}_p\right) \otimes \mathcal{C}^{\mathrm{sm}}\left(\widehat{\mathbb{Z}}^{p\times}, \mathbb{Z}_p\right)$$

(see also page 48 of Emerton's interpolation paper).

6.2. $\mathbb{G} = \mathrm{Res}_{\mathbb{Q}}^F\mathrm{GL}_1$ **where F is a number field.** We have

$$\mathrm{Res}_{\mathbb{Q}}^F\mathrm{GL}_1(\mathbb{R}) = (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \cong \left(\mathbb{R}^\times\right)^{r_1} \times \left(\mathbb{C}^\times\right)^{r_2} .$$

Thus $K_\infty^\circ = (S^1)^{r_2}$ and $A_\infty^\circ \cong \mathbb{R}_{>0}^\times$ for the embedding $\mathbb{R}_{>0}^\times \to (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$, $1 \mapsto 1 \otimes x$. This shows that

$$l_0 = r_1 + 2r_2 - r_2 - 1 = r_1 + r_2 - 1.$$

On the other hand we also see that $\mathrm{Res}_{\mathbb{Q}}^F\mathrm{GL}_1(\mathbb{R})/K_\infty^\circ\mathbb{R}_{>0}^\times$ is isomorphic to a finite union of $\mathbb{R}^{r_1+r_2-1}$. So $d = r_1 + r_2 - 1$ and $q_0 = 0$.

Now

$$(\mathbb{G}(\mathbb{Q})^\circ \cap K^p) \cap K_p = \mathbb{G}(\mathbb{Z})^\circ \cap K_p = \mathcal{O}_F^{\times,+} \cap K_p.$$

Note that

$$\mathbb{G}(\mathbb{Z}_p) = (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \cong \prod_v \mathcal{O}_{F_v}^\times,$$

so if $K_p \subset \mathbb{G}(\mathbb{Z}_p)$ is sufficiently small we have $K_p \cong \mathbb{Z}_p^{r_1+2r_2}$. On the other hand $\overline{\mathcal{O}_F^{\times,+} \cap K_p}$ be isomorphic to $\mathbb{Z}_p^{r_1+r_2-1-\delta}$, where δ is Leopoldt's defect. Thus

$$\widetilde{\mathrm{H}}_0(\mathbb{G}, K^p) \cong \mathbb{Z}_p\left[\left[\left(\overline{\mathcal{O}_F^{\times,+} \cap K_p}\backslash K_p\right)\right]\right]^{\Delta} \cong \mathbb{Z}_p\left[\left[\mathbb{Z}_p^{r_2+\delta}\right]\right]^{\Delta}.$$

This has codimension $r_1 + r_2 - 1 - \delta$, so we see it is equal to l_0 if and only if $\delta = 0$. So the conjecture of Calegari and Emerton in this case is equivalent to Leopoldt's conjecture.

6.3. $\mathbb{G} = \mathrm{Res}_{\mathbb{Q}}^F \mathrm{SL}_2$ **where F is a quadratic imaginary field.** We have

$$\mathbb{G}(\mathbb{R}) = \mathrm{Res}_{\mathbb{Q}}^F \mathrm{SL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{C})$$

so that $K_\infty = \mathrm{SU}_2(\mathbb{C})$ and $A_\infty^\circ = 1$, because $Z(\mathrm{Res}_{\mathbb{Q}}^F \mathrm{SL}_2) = \mathrm{Res}_{\mathbb{Q}}^F \mathrm{SL}_2 = \mathrm{Res}_{\mathbb{Q}}^F \mu_2$. It follows that $\dim G_\infty = 6$, $\dim K_\infty = 3$ so that $d = 3$. Moreover, the Cartan subgroup of $\mathrm{SL}_2(\mathbb{C})$ is the set of diagonal matrices in $M_2(\mathbb{C})$ with zero trace, while the Cartan subgroup of $\mathrm{SU}_2(\mathbb{C})$ is the set of matrices in $M_2(\mathbb{R})$ with trace zero. Thus $\mathrm{rank}(\mathrm{SL}_2(\mathbb{C})) = 2$ while $\mathrm{rank}\mathrm{SU}_2(\mathbb{C}) = 1$, so that $l_0 = 1$.

Let's figure out what one can say about the completed homology groups. First, by general principles, the homology $\widetilde{\mathrm{H}}_i$ vanishes for $i \geq 3$. Second, by a general theorem of Calegari and Emerton, one knows that the $\widetilde{\mathrm{H}}_i$ are all torsion if $\mathbb{G}$ is semisimple. Second, one knows that the spaces $Y(K_f)$ are all connected, because of strong approximation for semisimple simply connected and connected groups. Thus $\widetilde{\mathrm{H}}_0 = \mathbb{Z}_p$ (Lemma 4.1).

To say something about the completed homology groups, one uses the duality spectral sequences $E_2^{i,j} := \mathrm{Ext}^i(\widetilde{\mathrm{H}}_j, \mathbb{Z}_p[[K_p]]) \Rightarrow \widetilde{\mathrm{H}}^{\mathrm{BM}}_{3-i-j}$ and $E_2^{i,j} := \mathrm{Ext}^i(\widetilde{\mathrm{H}}^{\mathrm{BM}}_j, \mathbb{Z}_p[[K_p]]) \Rightarrow \widetilde{\mathrm{H}}_{3-i-j}$. One knows that $\mathrm{Ext}^0(\widetilde{\mathrm{H}}_j, \mathbb{Z}_p[[K_p]]) = \mathrm{Hom}\left(\widetilde{\mathrm{H}}_j, \mathbb{Z}_p[[K_p]]\right) = 0$ for all j, since the groups $\widetilde{\mathrm{H}}_j$ are torsion. Thus the leftmost column of the second page of the first spectral sequence vanishes. Moreover, since $\widetilde{\mathrm{H}}_0 = \mathbb{Z}_p$, we have that $\mathrm{Ext}^6(\widetilde{\mathrm{H}}_j, \mathbb{Z}_p[[K_p]]) = \mathbb{Z}_p$ and $\mathrm{Ext}^i(\widetilde{\mathrm{H}}_j, \mathbb{Z}_p[[K_p]]) = \mathbb{Z}_p$ for $0 \leq i \leq 5$, i.e. we have vanishing in the lowest row before the 6th column. Examining the second diagonal now shows that $\mathrm{Ext}^1\left(\widetilde{\mathrm{H}}_1, \mathbb{Z}_p[[K_p]]\right) \cong \widetilde{\mathrm{H}}^{\mathrm{BM}}_1$, and in particular, $\widetilde{\mathrm{H}}^{\mathrm{BM}}_1$ is torsion. On the other hand, by general principles $\widetilde{\mathrm{H}}^{\mathrm{BM}}_0 = 0$ since the symmetric space of $\mathrm{SL}_2(\mathbb{C})$ is not compact. So examining the second spectral sequence shows the entire lowest row vanishes, which implies $\mathrm{Hom}\left(\widetilde{\mathrm{H}}^{\mathrm{BM}}_1, \mathbb{Z}_p[[K_p]]\right) \cong \widetilde{\mathrm{H}}_2$, and as $\widetilde{\mathrm{H}}^{\mathrm{BM}}_1$ is torsion, we get $\widetilde{\mathrm{H}}_2 = 0$. So the second row in the first spectral sequence also vanishes. We will now explain why From examining the differentials now one sees that $\mathrm{Ext}^1(\widetilde{\mathrm{H}}_1, \mathbb{Z}_p[[K_p]]) \neq 0$; indeed, if not then we must have $\widetilde{\mathrm{H}}^{\mathrm{BM}}_1 = 0$, which would show the first row of the second spectral sequnce vanishes, and hence $\widetilde{\mathrm{H}}_1 \cong \mathrm{Hom}\left(\widetilde{\mathrm{H}}^{\mathrm{BM}}_2, \mathbb{Z}_p[[K_p]]\right)$, but $\widetilde{\mathrm{H}}^{\mathrm{BM}}_2 = 0$ by the first spectral sequence, so we get $\widetilde{\mathrm{H}}_1 = 0$; thus in the first spectral sequence, all rows above the lowest one are just 0, so from $\mathrm{Ext}^6(\widetilde{\mathrm{H}}_j, \mathbb{Z}_p[[K_p]]) = \mathbb{Z}_p$ one gets a contribution to $\widetilde{\mathrm{H}}^{\mathrm{BM}}_{-3}$ which doesn't make any sense. So this is a contradiction and we are finished.

To recap: $\widetilde{\mathrm{H}}_0 = \mathbb{Z}_p, \widetilde{\mathrm{H}}_1$ has codimension 1 and all the rest of the homology vanishes. This agrees with the prediction of Calegari and Emerton.

7. The case of GL_2 and local global compatibility

The case GL_2 is of a lot of interest because in this case the symmetric spaces are modular curves. In particular the completed cohomology carries around an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

First of all, we note that $l_0 = 0$, $q_0 = 1$. The completed cohomology vanishes except in degrees $0, 1$. In degree 0, it is not hard to check that the determinant map implies an isomorphism $\widetilde{\mathrm{H}}^0(\mathrm{GL}_2, K^p) \cong \widetilde{\mathrm{H}}^0(\mathrm{GL}_2, \det(K^p))$, and as such it is isomorphic to $\mathcal{C}\left(\Delta \times \mathbb{Z}_p^\times, \mathbb{Z}_p\right)$ for some finite set Δ. See 7.2.11 in Emerton's "dedicated to Coates"' paper for the actions (similar to the case of GL_1 but also have a Galois action now: basically it's the same but now the Galois group acts in the expected way too).

The really interesting thing happens in degree 1. Fix a maximal non Eisenstein ideal m in the Hecke algebra $\mathbb{T}$ (we are implicitly fixing a tame level, which determines the Hecke algebra). One has an associated p-adic Galois representation $\overline{\rho} = \overline{\rho}_m$ of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. (To make what follows absolutely true one probably wants to assume more conditions, but I will ignore this). Then it seems like one has a result of the following form

$$\widetilde{\mathrm{H}}^1(\mathrm{GL}_2) \cong \rho^{\mathrm{univ}} \otimes_{\mathbb{T}_m} \pi_p\left(\rho^{\mathrm{univ}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}\right) \underset{l\neq p\,\mathbb{T}_m}{\otimes'} \pi_l\left(\rho^{\mathrm{univ}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)}\right),$$

(giving a decomposition of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \times \mathrm{GL}_2(\mathbb{Q}_p) \otimes \mathrm{GL}_2(\mathbb{A}_f)$ representations) where we have implicitly identified $\mathbb{T}_m$ and $R_{\overline{\rho}}$ via an $R = T$ theorem; here is $\pi_p\left(\rho^{\mathrm{univ}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}\right)$ the Banach space representation attached to $\rho^{\mathrm{univ}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ by the p-adic Langlands correspondence while $\pi_l\left(\rho^{\mathrm{univ}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)}\right)$ is associated to $\rho^{\mathrm{univ}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)}$ by the local Langlands correspondence (note however there is some caveat that this is not exactly the definition of the π_l in general for some reason). (I feel also like the action of G_∞ should also appear here somewhere via conjugation).

In the case where $K^p = 1$, Emerton says in 12:40 of his youtube talk that

$$\widetilde{\mathrm{H}}^1(\mathrm{GL}_2) \cong \rho^{\mathrm{univ}} \otimes_{\mathbb{T}_m} \pi_p\left(\rho^{\mathrm{univ}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}\right) \otimes \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{T}_m, \mathbb{Z}_p)$$

so we get something easier.

The issue with the modification of π_l is explained very well in Emerton's local global compatibility conjecture paper, section 2.1.1. Namely, $\pi_l(\sigma)$ is just the local langlands correspondence if σ is such that the local langlands thing is generic (i.e. admits a Whittaker model, i.e. is infinite dimensional). In the remaining case (i.e. exactly when $LLC(\sigma)$ is a character composed with the determinant) one modifies this so that $\pi_l(\sigma)$ is the induction for which the $LLC(\sigma)$ is a subquotient (i.e. an extension of this character and the twist of a Steinberg). So it is some kind of a closure of LLC so that dimensions of LLC don't go completely insane (drop from ifninity to 1 at a point) so they can live in families. Note that one has to use the Tate normalization of LLC for this entire story. Finally, we note that if f is a classical cuspidal newform, then all the π_l's are generic so it's just the same as taking LLC (see remark 7.1.2 of Emerton's a local global compatibility conjecture paper).

Really, it seems this statement is probably more of a conjecture than an actual theorem, because we are taking here the p-adic Langlands correspondence and the local langlands correspondence in families, which may or may not exist. But this is the kind of theorem mentioned by Emerton in his talk. See also the introduction to Emerton's local global compatibility paper.

In particular, this implies a theorem of the following form. For each ρ a representation into $\overline{\mathbb{Q}}_p$ reducing to ρ, one should have

$$\mathrm{Hom}_{\mathbb{Q}_p\left[\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)\right]}\left(\rho, \widetilde{\mathrm{H}}^1\left(\mathrm{GL}_2\right)\right) \cong \pi_p\left(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}\right) \bigotimes_{l\neq p_{\mathbb{T}_m}}{}' \pi_l\left(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)}\right).$$

This is actually a theorem (Theorem 1.2.1 of Emerton).