# A CAYLEY-HAMILTON TYPE THEOREM

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We present a theorem which is a generalization of the Cayley-Hamilton theorem for the case of an invertible matrix. What is interesting about this theorem is that although its description involves only linear algebra, it can be naturally proved with commutative algebra!

**Theorem.** *Let $V$ be a finite dimensional vector space over a field $k$. If $A_1, ..., A_n$ pairwise commute and $A_1V + ... + A_nV = V$, then there is a polynomial $P \in k[x_1, ..., x_n]$ with zero lower coefficient such that $P(A_1, ..., A_n) = I$.*

Notice that this theorem is only interesting when none of the matrices are invertible, for otherwise it follows immediately from the Cayley-Hamilton theorem.

*Proof.* Let $k[x_1, ..., x_n]$ act on $V$ by $x_iv = A_iv$. Because the matrices pairwise commute, this gives $V$ the structure of a $k[x_1, ..., x_n]$-module. Let $J = (x_1, ..., x_n)$, the ideal in $k[x_1, ..., x_n]$. We have $V = A_1V + ... + A_nV \subseteq JV$, hence $JV = V$; by Nakayama's Lemma (cf [1], Corollary 2.5), this means there exists a polynomial $h = 1 - P$ with $P \in J$ such that $hV = 0$. For $v \in V$ we have $0 = hv = (I - P(A_1, ..., A_n))v$, hence $P(A_1, ..., A_n) = I$.

**Remark.** The statement of the theorem looks very similar to the weak version of Hilbert's Nullstellensatz. It would be interesting to know if there is some similar argument which proves it directly from Nakayama's Lemma as in the above proof.

## References

[1] M. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Massachusetts : Addison-Wesley Publishing, 1969.