BASIC EXAMPLES AND COMPUTATIONS IN p-ADIC HODGE THEORY

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Throught, the convention is that the cyclotomic character had Hodge-Tate weight 1. We always denote by V a p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, which by default has coefficients in $\mathbb{Q}_p$.

1. Examples and Counterexamples

1.1. **Hodge-Tate but not de Rham.** It is well known that a non trivial extension

$$0 \to \mathbb{Q}_p(-1) \to V \to \mathbb{Q}_p \to 0$$

must be split if it is de Rham. This is because any such extension which is de Rham is also crystalline, by comparing dimensions of H^1_g and $\mathrm{H}^1_{\mathrm{st}}$. A computation I shall produce later shows that the dimension of Ext in the latter is 0. So any nonsplit extension will give a non de Rham extension, but it will be Hodge-Tate, by 2.2. To see there is such a nonsplit extension, one must show $\dim \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}_p}, \mathbb{Q}_p(-1)) > 0$. This follows immediately from the Euler characteristic formula.

Another way to see this is to calculate the dimension of the cohomology group of de Rham extensions $\mathrm{H}^1_g(G_{\mathbb{Q}_p}, \mathbb{Q}_p(-1))$ and to see it is zero-dimensional.

1.2. **de Rham but not Semistable.** In dimension 1, being semistable is the same as being crystalline, so it suffices to find a 1-dimensional example which is de Rham but not crystalline. Any ramified character with finite image will do.

1.3. **Semistable but not Crystalline.** Take a semistable elliptic curve over $\mathbb{Q}_p$, say. By Tate's theorem it is $G_{\mathbb{Q}_p}$ isomorphic to $\overline{\mathbb{Q}}_p/q^{\mathbb{Z}}$ for some $q \in p\mathbb{Z}_p$. Therefore, it is not too hard to calculate the Galois representation is an extension

$$0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0,$$

for which one can explicitly work out the following invariants (calculations in my notes or in Berger's notes), for $\mathrm{D} = \mathrm{D}_{\mathrm{st}}(V)$:

1. D is two dimensional (so V is semistable), with basis $x = t^{-1} \otimes \varprojlim \zeta_{p^n}$, $y = -\log_p[q^\flat]t^{-1} \otimes \varprojlim \zeta_{p^n} + 1 \otimes \varprojlim q^{1/p^n}$.

2. It has the filtration

$$\mathrm{Fil}^i(\mathrm{D}) = \begin{cases} \mathrm{D} & i \leq -1 \\ \mathrm{span}\left\{\log_p q \cdot x + y\right\} & i = 0 \\ 0 & i \geq 1 \end{cases}.$$

3. The action of φ is given by $\varphi(x) = p^{-1}x$, $\varphi(y) = y$, and

4. The action of N is given by $N(x) = 0$, $N(y) = v_p(q)x$.

In particular, we have $\mathrm{D_{cris}}(V) = \mathrm{D_{st}}(V)^{N=0} = \mathrm{span}\,\{x\}$ is 1-dimensional, so V is not crystalline.

1.4. **Extension of Hodge-Tate which is not Hodge-Tate.** Take an extension

$$0 \to \mathbb{Q}_p \to V \to \mathbb{Q}_p \to 0,$$

for which $G_{\mathbb{Q}_p}$ acts by

$$\begin{pmatrix} 1 & \log_p \chi(g) \\ 0 & 1 \end{pmatrix}.$$

Then the Sen operator is by definition given by $\frac{\log_p \gamma}{\log_p \chi(\gamma)}$, which is just

$$\Theta_V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which is not semisimple.

1.5. **Extension of de Rham which is not de Rham, but is Hodge-Tate.** Same as 1.1.

1.6. **Extensions of Semistables which is not Semistable, but is de Rham.** I couldn't come up with an example, but apparently this can never happen, by a theorem of Hyodo.

1.7. **Extensions of Crystallines which is not Crystalline, but is Semistable.** Same example as in 1.3.

1.8. **de Rham representation with $\mathrm{D_{st}} = 0$.** Same example as in 1.2.

1.9. **Example of a de Rham representation which does not come from geometry.** Just take an unramified character whose value on Frobenius isn't an algebraic number.

2. General Facts

2.1. **Nonpositive Hodge-Tate weights implies $\mathrm{D}^+_{\mathrm{dR}} = \mathrm{D_{dR}}$.** For $n \geq 1$, we have the exact sequence

$$0 \to t^{-n+1}\mathrm{B}^+_{\mathrm{dR}} \to t^{-n}\mathrm{B}^+_{\mathrm{dR}} \to \mathbb{C}_p(-n) \to 0,$$

tensoring with V and taking cohomology, using Tate's theorem, shows that

$$\mathrm{H}^0(\mathrm{G}_{\mathbb{Q}_p}, t^{-n+1}\mathrm{B}^+_{\mathrm{dR}} \otimes V) \cong \mathrm{H}^0(\mathrm{G}_{\mathbb{Q}_p}, t^{-n}\mathrm{B}^+_{\mathrm{dR}} \otimes V),$$

and taking the limit over n, we see all of these groups are isomorphic to $\mathrm{D_{dR}}(V)$. In particular taking $n = 1$ we see that $\mathrm{D_{dR}}(V) = \mathrm{D}^+_{\mathrm{dR}}(V)$. □

2.2. **Extension of Hodge-Tate representations with different Hodge-Tate weight is Hodge-Tate.** Let $0 \to U \to V \to W \to 0$ be an extension with U, W Hodge-Tate, with different Hodge-Tate weights. Then the Sen polynomials of the operators Θ_U and Θ_W are mutually coprime. As each of these are diagonaizable by hypothesis, the Sen operator Θ_V which has blocks Θ_U and Θ_W is also diagonaizable. □

2.3. **Extension of de Rham representations with strictly decreasing Hodge-Tate weights is de Rham.** Let $0 \to U \to V \to W \to 0$ be an extension with U, W de Rham. Without loss of generality, we may assume U has Hodge-Tate weight ≥ 1, and W has Hodge-Tate weights ≤ 0. Taking D_{dR}, it suffices to show $\mathrm{D}_{\mathrm{dR}}(V) \to \mathrm{D}_{\mathrm{dR}}(W)$ is surjective. Since by 2.1 $\mathrm{D}_{\mathrm{dR}}(W) = \mathrm{D}^+_{\mathrm{dR}}(W)$, it suffices to prove that $\mathrm{D}^+_{\mathrm{dR}}(V) \to \mathrm{D}^+_{\mathrm{dR}}(W)$ is surjective, and hence it is enough to prove $\mathrm{H}^1(G_{\mathbb{Q}_p}, \mathrm{B}^+_{\mathrm{dR}} \otimes U) = 0$. Indeed, for each $n \geq 0$ we have an exact sequence

$$0 \to t^{n+1}\mathrm{B}^+_{\mathrm{dR}} \to t^n\mathrm{B}^+_{\mathrm{dR}} \to \mathbb{C}_p(n) \to 0,$$

and Tate's theorem implies that $\mathrm{H}^1(G_{\mathbb{Q}_p}, t^{n+1}\mathrm{B}^+_{\mathrm{dR}} \otimes U) \cong \mathrm{H}^1(G_{\mathbb{Q}_p}, t^n\mathrm{B}^+_{\mathrm{dR}} \otimes U)$. By succesive approximation, $\mathrm{H}^1(G_{\mathbb{Q}_p}, \mathrm{B}^+_{\mathrm{dR}} \otimes U) = 0$. $\square$

2.4. **Extension of Crystalline/Semistable representations with far enough Hodge-Tate weights which are decreasing is Crystalline/Semistable.** For this we need to first carry out a general study of properties of extensions. The only properties we are using here is that this is an exact tensorial category, and that the natural map $1 \to Y \otimes Y^\vee$ has the natural projection $\frac{1}{n}\mathrm{Tr} : Y \otimes Y^\vee \to 1$,.

Lemma. *There is a canonical isomorphism* $\mathrm{Ext}(X, Y) \cong \mathrm{Ext}(X \otimes Y^\vee, 1) \cong \mathrm{H}^1(X \otimes Y^\vee)$. *The second correspondence is standard.*

The first correspondence is given by

$$0 \to X \to E \to Y \to 0$$

mapsto to the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \otimes Y^\vee & \longrightarrow & E \otimes Y^\vee & \longrightarrow & Y \otimes Y^\vee & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & X \otimes Y^\vee & \longrightarrow & (E \otimes Y^\vee) \times_{X \otimes Y^\vee} 1 & \longrightarrow & 1 & \longrightarrow & 0 \end{array},$$

and in the other direction

$$0 \to X \otimes Y^\vee \to F \to 1 \to 0$$

mapsto to the pushforward diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \otimes Y^\vee \otimes Y & \longrightarrow & F \otimes Y & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \frac{1}{n}\mathrm{Tr}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & (F \otimes Y) \amalg_{X \otimes Y^\vee \otimes Y} X & \longrightarrow & Y & \longrightarrow & 0 \end{array}.$$

Proof. The constructions are canonical, so maps between $\mathrm{Ext}(X, Y)$ and $\mathrm{Ext}(X \otimes Y^\vee, 1)$ are induced in both directions. It remains to check they are inverses.

Write φ for the first map and ψ for the second. Given an extension E, we get the extension $\psi(\varphi(E))$. By definition it is the colimit of $\varphi(E) \otimes Y$ and X over $X \otimes Y^\vee \otimes Y$. To give a map from this colimit to E we need to give maps from X and $\varphi(E)$ to E which agree on the image of $X \otimes Y^\vee \otimes Y$. It is clear how to give a map from X. To give a map from $\varphi(E) \otimes Y$ to E, note that by construction we have a map from $\varphi(E)$ to $E \otimes Y^\vee$, so by composing with the averaged trace map we get a map from $\varphi(E) \otimes Y$ to E. If we begin with an element of $X \otimes Y \otimes Y^\vee$ it is clear that both maps into E give us the map $1 \otimes \frac{1}{n}\mathrm{Tr}$. Thus we get a unique map $\psi(\varphi(E)) \to E$ commuting with the maps from X and from $\varphi \otimes Y$. We claim this map also induces a the identity map on Y. Indeed each element of Y has an inverse in $\varphi(E) \otimes Y$, which naturally maps to E and then to Y. This map is the identity, and we know maps comping from $\varphi(E) \otimes Y$ are going to commute (just write diagram). So the map $\psi(\varphi(E)) \to E$ is induced and it induces the identity on X and on Y, which means it is the same extension.

Dualising the above argument gives the other direction of the argument. This completes the proof. □

Corollary 2.1. *Let X and Y both be $*$, where $* \in \{$crystalline, semistable, de Rham or Hodge Tate$\}$. Then an extension $0 \to X \to E \to Y \to 0$ has the propery $*$ if and only if the corresponding extension $0 \to X \otimes Y^\vee \to F \to 1 \to 0$ has the property $*$.*

Proof. Suppose $0 \to X \to E \to Y \to 0$ has the property $*$. Then $0 \to X \otimes Y^\vee \to E \otimes Y^\vee \to Y \otimes Y^\vee \to 0$ also has $*$. Then F is the by constrution the fiber of the map $1 \to Y \otimes Y^\vee$, so it injects into $E \otimes Y^\vee$, so it has property $*$.

Conversely suppose $0 \to X \otimes Y^\vee \to F \to 1 \to 0$ has property $*$. Then $0 \to X \otimes Y^\vee \otimes Y \to F \otimes Y \to Y \to 0$ also has $*$. Since E is by definition the pushforward of the map $Y \otimes Y^\vee \to 1$, it is surjected upon by $F \otimes Y$, so it is has property $*$. □

This reduces all questions about properties of extensions to the case of an extension by the trivial representation. For instance, this reduces the proof in 2.3 to the case where W is trivial, which makes it easier.

The easiest thing to do now, since the de Rham question is settled, is to compare the dimensions as in section 2.7, and be able to deduce directily when extensions are automatically crystalline when Hodge-Tate weights are far anourhg apart (namely one wants something like $\mathrm{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1} = 0$ for $V = X \otimes Y^\vee$, and this is automatic for large enough HT-weights by comparing the Newton and Hodge polygons.

2.5. **Every Hodge-Tate representation has $\mathrm{D}_{\mathrm{dR}} \neq 0$.** *Proof.* Without loss of generality, we may assume 0 is the minimal Hodge-Tate weight of V. It now suffices to show that $\mathrm{D}^+_{\mathrm{dR}}(V) \neq 0$. By Tate's theorem, we have $\mathrm{H}^0(\mathbb{Q}_p, \mathbb{C}_p \otimes V) \neq 0$ and $\mathrm{H}^i(\mathbb{Q}_p, \mathbb{C}_p(n) \otimes V) = 0$ for $n > 0$, $i = 0, 1$. For each $n \geq 0$, we have an exact sequence

$$0 \to t^{n+1}\mathrm{B}^+_{\mathrm{dR}} \to t^n\mathrm{B}^+_{\mathrm{dR}} \to \mathbb{C}_p(n) \to 0.$$

For $n > 0$, this implies that $\mathrm{H}^i(\mathbb{Q}_p, t^{n+1}\mathrm{B}^+_{\mathrm{dR}} \otimes V) \cong \mathrm{H}^i(\mathbb{Q}_p, t^n\mathrm{B}^+_{\mathrm{dR}} \otimes V)$ for $i = 0, 1$. By continuity, this implies $\mathrm{H}^i(\mathbb{Q}_p, t^n\mathrm{B}^+_{\mathrm{dR}} \otimes V) = 0$ for each $n > 0$, in particular for $n = 1$. Now tensoring with V and taking cohomology in the above sequence for $n = 0$, the vanishing of cohomology above implies that

$$\mathrm{D}^+_{\mathrm{dR}}(V) \cong \mathrm{H}^0(\mathbb{Q}_p, \mathbb{C}_p \otimes V) \neq 0.$$

□

2.6. **Fundamental exacts sequences and consequences.** One has two fundamental exact sequences:

$$0 \to \mathbb{Q}_p \to \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \to \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+} \to 0$$

$$0 \to \mathbb{Q}_p \to \mathrm{Fil}^0\mathrm{B}_{\mathrm{cris}} \xrightarrow{1-\varphi} \mathrm{B}_{\mathrm{cris}} \to 0.$$

A related exact sequence is

$$0 \to \mathbb{Q}_p \to \mathrm{B}_{\mathrm{cris}} \oplus \mathrm{B}_{\mathrm{dR}}^{+} \xrightarrow{(1-\varphi,x-y)} \mathrm{B}_{\mathrm{cris}} \oplus \mathrm{B}_{\mathrm{dR}} \to 0.$$

It follows from the first two as follows. The only nontrivial thing is the surjectivity of the last map. Well, if (a,b) is in $\mathrm{B}_{\mathrm{cris}} \oplus \mathrm{B}_{\mathrm{dR}}$, take $c \in \mathrm{Fil}^0\mathrm{B}_{\mathrm{cris}}$ such that $(1-\varphi)(c) = a$, which exists by the second sequence. Now by the first sequence, there exists a $d \in \mathrm{B}_{\mathrm{cris}}^{\varphi=1}$ such that $b - d \in \mathrm{B}_{\mathrm{dR}}^{+}$. Then $(c+d, c+d-b)$ is in $\mathrm{B}_{\mathrm{cris}} \oplus \mathrm{B}_{\mathrm{dR}}^{+}$ and maps to (a,b).
From this we have a few direct consequences.

2.6.1. *Dimension formulas.*

Lemma 2.1. *If V is de Rham then* $\left(\mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} V\right)^{G_K} = \mathrm{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V))$.

Proof. One has the exact sequence

$$0 \to \mathrm{B}_{\mathrm{dR}}^{+} \to \mathrm{B}_{\mathrm{dR}} \to \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+} \to 0.$$

Tensoring with V and taking G_K invariants, we have

$$0 \to \mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V)) \to \mathrm{D}_{\mathrm{dR}}(V) \to \left(\mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} V\right)^{G_K},$$

so it suffices to check the dimensions match. Indeed, we have

$$\begin{aligned} \dim_{\mathbb{Q}_p} V = \dim_K \mathrm{D}_{\mathrm{dR}}(V) &\le \dim_K \mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V)) + \dim_K \left(\mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} V\right)^{G_K} \\ &\le \sum_{i \in \mathbb{Z}} \dim_K \mathrm{H}^0(\mathbb{C}_p(i) \otimes V) = \dim_{\mathbb{Q}_p} V, \end{aligned}$$

where the first equality holds because V is de Rham, and the last equality holds because V is Hodge-Tate. For the middle equality, one only needs to use $t^i\mathrm{B}_{\mathrm{dR}}^{+}/t^{i+1}\mathrm{B}_{\mathrm{dR}}^{+} = \mathbb{C}_p(i)$. ☐

Suppose then that V is de Rham, and tensor

$$0 \to \mathbb{Q}_p \to \mathrm{B}_{\mathrm{cris}} \xrightarrow{(1-\varphi,x)} \mathrm{B}_{\mathrm{cris}} \oplus \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+} \to 0$$

with V, then take Galois invariants (the exactness of this sequence follows from the third sequence above). One gets

$$0 \to V^{G_K} \to \mathrm{D}_{\mathrm{cris}}(V) \to \mathrm{D}_{\mathrm{cris}}(V) \oplus \mathrm{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V)) \to \mathrm{H}^1_f(K,V) \to 0.$$

From this one gets the formula

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(K,V) = [K:\mathbb{Q}_p]\left(\dim_{\mathbb{Q}_p} V - \dim_K \mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V))\right) + \dim_{\mathbb{Q}_p} V^{G_K}.$$

To calculate the formula for $\mathrm{H}^1_e(K,V)$, one proceeds similarly. Namely, one uses the first sequence, and gets

$$0 \to V^{G_K} \to (V \otimes \mathrm{B}_{\mathrm{cris}}^{\varphi=1})^{G_K} \to \mathrm{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0(\mathrm{D}_{\mathrm{dR}}(V)) \to \mathrm{H}^1_e(K,V) \to 0,$$

showing (as $(V \otimes \mathrm{B}_{\mathrm{cris}}^{\varphi=1})^{G_K} = \mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1}$) that

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_e(K,V) = \dim_{\mathbb{Q}_p} \mathrm{H}^1_f(K,V) - \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1}.$$

Finally, to compute $\dim_{\mathbb{Q}_p} \mathrm{H}^1_g(K,V)$, we note that by the theorem of Bloch and Kato, $\mathrm{H}^1_e(K,V)$ and $\mathrm{H}^1_g(K,V^*(1))$ are exact annihilators. Thus $\dim_{\mathbb{Q}_p} \mathrm{H}^1_e(K,V)+\dim_{\mathbb{Q}_p} \mathrm{H}^1_g(K,V^*(1)) = \dim_{\mathbb{Q}_p} \mathrm{H}^1(K,V)$, and

$$\begin{aligned}\dim \mathrm{H}^1_g(K,V) &= \dim_{\mathbb{Q}_p} \mathrm{H}^1(K,V) - \dim_{\mathbb{Q}_p} \mathrm{H}^1_e(K,V^*(1)) \\ &= \dim_{\mathbb{Q}_p} \mathrm{H}^1(K,V) - \dim_{\mathbb{Q}_p} \mathrm{H}^1_f(K,V^*(1)) + \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1} \\ &= \dim_{\mathbb{Q}_p} \mathrm{H}^1_f(K,V) + \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1}.\end{aligned}$$

2.7. **Induction and Restriction.** In this section we shall understand the relation between admissibility of a representation and its induction/restriction, as well as the relation between their Hodge-Tate weights.

Here we shall always assume $[G:H] < \infty$, where G and H are absolute Galois group of finite extensions of $\mathbb{Q}_p$.

Lemma 2.2. *Restriction and Induction commutes with duals.*

Proof. It is clear that restriction commutes with duals. This now follows for induction, but the following formal computation and the Yoneda lemma:

$$\begin{aligned}\mathrm{Hom}_G(\mathrm{Ind}_H^G(V^*), W) &\cong \mathrm{Hom}_H(V^*, \mathrm{Res}_H^G W) \cong \mathrm{Hom}_H(\left(\mathrm{Res}_H^G W\right)^*, V) \\ &\cong \mathrm{Hom}_H(\mathrm{Res}_H^G W^*, V) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G W^*, V) \cong \mathrm{Hom}_G(W^*, \mathrm{Ind}_H^G V) \\ &\cong \mathrm{Hom}_G(\left(\mathrm{Ind}_H^G(V)\right)^*, W).\end{aligned}$$

For the second to last isomorphism we use the fact that the index is finite, so that induction and restriction are adjoints in both directions. □

Now let B be one of Fontaine's rings. Recall being admissible is having the correct dimension over B^G.

Proposition 2.1. *1. Suppose V is (B,H)-admissible. Then $\mathrm{Ind}_H^G V$ is G-admissible if and only if $[B^H : B^G] = [G:H]$.*
2. Suppose V is (B,G) admissible. Then so is $\mathrm{Res}_H^G V$.

Proof. 1. We have, by the lemma

$$D_B^G(\mathrm{Ind}_H^G V^*) = \mathrm{Hom}_G(\mathrm{Ind}_H^G V, B) \cong \mathrm{Hom}_H(V,B) = D_B^H(V),$$

so

$$\dim_{B^G} D_B^G(\mathrm{Ind}_H^G V^*) = \dim_{B^G} D_B^H(V) = [B^H : B^G] \dim_{B^H} D_B^H(V) = [B^H : B^G] \dim V.$$

On the other hand

$$[G:H] \dim V = \dim \mathrm{Ind}_H^G V.$$

2. We have an injection

$$\mathrm{Hom}(V,B)^G \otimes_{B^G} B^H \hookrightarrow \mathrm{Hom}(V,B)^H.$$

Taking dimensions, we see that

$$\dim V = \dim_{B^G} \mathrm{Hom}(V,B)^G = \dim_{B^H} \mathrm{Hom}(V,B)^G \otimes_{B^G} B^H \leq \dim_{B^H} \mathrm{Hom}(V,B)^H \leq \dim V.$$

Hence all the inequalities are equalities. □

Corollary 2.2. *Let L/K be local fields with Galois groups G_K, G_L.*

1. Suppose V is a representation of G_K over $\mathbb{Q}_p$, which is Hodge-Tate. Then $\mathrm{Res}^{G_K}_{G_L} V$ is also Hodge-Tate with the same weights.

2. Suppose V is a representation of G_L over $\mathbb{Q}_p$, which is Hodge-Tate. Then $\mathrm{Ind}^{G_K}_{G_L} V$ is also Hodge-Tate with the same weights, each appearing $[L:K]$ times.

Proof. 1. The previous proposition shows that $\mathrm{Res}^{G_K}_{G_L} V$ is Hodge-Tate. To show it has the right Hodge-Tate weights, notice that the argument of the proposition implies (by comparing dimensions, using they are both Hodge-Tate) that

$$\mathrm{Hom}_{G_K}(V, \mathbb{C}_p(i)) \otimes_K L \cong \mathrm{Hom}_{G_L}(V, \mathbb{C}_p(i))$$

as L-vector spaces, for each i.

2. This follows from the proposition above and from the isomorphism

$$\mathrm{Hom}_{G_K}(\mathrm{Ind}^{G_K}_{G_L}(V), \mathbb{C}_p(i)) \cong \mathrm{Hom}_{G_L}(V, \mathbb{C}_p(i)).$$

□

2.8. **Hodge-Tate representations with only one type of weight.** For the following, let V be a Hodge-Tate representation. One has the following theorem of Sen (corollary to Theorem 11 of the continuous cohomology paper).

Theorem 2.1. *Let V be a representation all of whose Hodge-Tate weights are the same. Then V is potentially unramified up to a Tate twist.*

Corollary 2.3. *Let V is a representation all of whose Hodge-Tate weights are the same. Then it is de Rham.*

We want to investigate what happens in the crystalline case. For this we shall prove the following lemma.

Lemma 2.3. *Let $G = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p) \cong \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$. Then $\mathrm{H}^1(G, \mathrm{GL}_n(\breve{\mathbb{Z}}_p)) = 0$.*

Proof. First, notice that we have

$$\mathrm{H}^1(G, \mathrm{GL}_n(\overline{\mathbb{F}}_p)) = 0$$

by Hilbert's theorem 90, and

$$\mathrm{H}^1(G, \mathrm{M}_n(\overline{\mathbb{F}}_p)) = 0,$$

by the Artin-Scherier sequence. Namely, in this case this follows from Local Fields, chapter XIII, proposition 1, once we notice $\sigma - 1 : \mathrm{M}_n(\overline{\mathbb{F}}_p) \to \mathrm{M}_n(\overline{\mathbb{F}}_p)$ is surjective.

Now consider the following exact sequence:

$$1 \to 1 + p\mathrm{M}_n(\breve{\mathbb{Z}}_p) \to \mathrm{GL}_n(\breve{\mathbb{Z}}_p) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p) \to 1.$$

Taking cohomology, the desired vanishing of $\mathrm{H}^1(G, \mathrm{GL}_n(\breve{\mathbb{Z}}_p))$ will vanish from that of $\mathrm{H}^1(G, 1+p\mathrm{M}_n(\breve{\mathbb{Z}}_p)) \cong \mathrm{H}^1(G, \mathrm{M}_n(\breve{\mathbb{Z}}_p))$, since $\mathrm{H}^1(G, \mathrm{GL}_n(\overline{\mathbb{F}}_p)) = 0$. But the vanishing of $\mathrm{H}^1(G, \mathrm{M}_n(\breve{\mathbb{Z}}_p))$ follows from that of $\mathrm{H}^1(G, \mathrm{M}_n(\overline{\mathbb{F}}_p))$ by succesive approximation. □

Corollary 2.4. *Let V is a representation all of whose Hodge-Tate weights are the same. The following are equivalent:*

1. V is crystalline.

2. V is semistable.

3. V is unramified.

Proof. We may twist and assume all the Hodge-Tate weights are 0.

First, we claim that semistable implies crystalline. Indeed, the monodromy operator on the associated filtered modules take Hodge-Tate weight n-part to the $n-1$ part, so it has to be trivial, so the representation is crystalline.

Next, we claim crystalline implies unramified. Indeed, consider the associated filtered φ-module D. Then the associated representation is $\mathrm{Fil}\,(B_{\mathrm{cris}} \otimes_{K_0} D)^{\varphi=1} = (B_{\mathrm{cris}} \otimes_{K_0} D)^{\varphi=1}$. This has $\mathbb{Q}_p$-dimension at most $\dim_{K_0} D$, so if we show that the subspace $\left(\breve{\mathbb{Q}}_p \otimes_{K_0} D\right)^{\varphi=1}$ has dimension $\dim_{K_0} D$, they will have to be equal, and then the claim will be true because $\left(\breve{\mathbb{Q}}_p \otimes_{K_0} D\right)^{\varphi=1}$ is obviously unramified. To show this, note that $\breve{\mathbb{Q}}_p \otimes_{K_0} D$ is non other than a $\breve{\mathbb{Q}}_p$-vector space with a frobenius semilinear action. Since all the slopes of the Hodge polygon are 0, the same holds for the Newton polygon, so this actually descends to a module over $\breve{\mathbb{Z}}_p$. Then by the lemma, this action has a fixed basis, and we're done.

Finally suppose V is unramified. Then Frob_p acts on V through some matrix of $\mathrm{GL}_n(\mathbb{Z}_p)$. Then $\left(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$ contains $\left(\breve{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} V\right)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/K_0)}$, which has the right dimension by the lemma. So V is crystalline. □