DEFORMATION THEORY

GAL PORAT

Abstract. This note is a basic summary of certain results in deformation theory. The focus is on the deformation theory of curves and nonsingular varieties, with applications to deformation of modular curves in mind.

Let k be a perfect field of characteristic p and let X_0/k be a scheme or variety. We are interested in the question of how it can be lifted or deformed to either a formal scheme over $\mathrm{Spf}W(k)$ or a flat variety over $\mathrm{Spec}W(k)$. Similarly we will be interested in the question of lifting a line bundle over X_0 once a lift of X_0 has been chosen. Note that this can also be done for vector bundles, but we will not be concerned with that here.

Some of the things in the next section might be initially small variations of what is actually proved in the reference I have given. I will thus mark them by a warning label **(!)**.

1. Lifting along Artinian rings

Following [Ha, section 2.10] we let C be an Artinian ring with $C/m_C = k$. A deformation of X_0 to an Artinian ring means a scheme X/C, flat over C, together with a closed immersion $X_0 \hookrightarrow X$ which induces $X_0 \xrightarrow{\sim} X \times_C k$.

Now let C' be another Artinian ring over k together with a map $C' \to C$. We can ask to extend X to X' along $C' \to C$, up to a suitable notion of equivalence.

Write $J = \ker(C' \to C)$, and suppose that J is killed by $m_{C'}$, so that it is a vector space over k. For instance, one could take something like $C = k[\varepsilon]/\varepsilon^n$ and $C' = k[\varepsilon]/\varepsilon^{n+1}$. Moreover, let $\mathcal{T}_{X_0}$ be the tangent sheaf of X_0. It is a coherent sheaf on X_0, and if X_0 is nonsingular it is a vector bundle.

The following theorem gives a critertion for when we can deform from C to C'.

Theorem 1.1. *(Corollary 10.3 of* [Ha]*) Suppose X_0 is nonsingular, and let X be a deformation of X_0 to C.*

(a) There is an obstruction class in $\mathrm{H}^2(X_0, \mathcal{T}_{X_0} \otimes_k J) = \mathrm{H}^2(X_0, \mathcal{T}_{X_0}) \otimes_k J$ which vanishes if and only if X can be extended to X' over C'.

(b) If such an extension exists, their equivalence class forms a torsor under $\mathrm{H}^1(X_0, \mathcal{T}_{X_0} \otimes_k J) = \mathrm{H}^1(X_0, \mathcal{T}_{X_0}) \otimes_k J$.

One way to think about this is that $\mathrm{H}^1(X_0, \mathcal{T}_{X_0} \otimes_k J)$ gives a tangent space in which one may hope to deform and $\mathrm{H}^2(X_0, \mathcal{T}_{X_0} \otimes_k J)$ gives the obstructions in this tangent space to deforming. Note also that if X_0 "lives in a moduli space", $\mathrm{H}^1(X_0, \mathcal{T}_{X_0})$ is the tangent space of that moduli space at the point corresponding to X_0. More on this later when we talk about universal deformations.

Remark 1.1. There is also a more general theorem which applies even if X_0 is singular. This has to do with a derived version of the tangent sheaf $\mathcal{T}_{X_0}$ via André-Quillen cohomology. But we're mainly concerned here with nonsingular varieties.

Example 1.1. 1. If X_0/k is affine, it can always be deformed uniquely from X to X', because higher coherent cohomology groups of affine schemes vanish.

2. If X_0/k is a projective curve, by dimension vanishing H^2 is always 0 for any coherent sheaf, so it can always be deformed from X to X', but usually not uniquely. Indeed, $\mathrm{H}^1(X_0, \mathcal{T}_{X_0})$ has dimension 0 for $\mathbb{P}^1_k$, dimension 1 for elliptic curves and dimension $3g-3$ for genus $g \geq 2$.

3. If in particular we take $X_0 = \mathbb{P}^1_k$ then the relevant H^1 also vanishes. So $\mathbb{P}^1_k$ is *rigid*: it deforms uniquely (as if it were formally etale over k, which it is not, since $\Omega^1_{\mathbb{P}^1_k/k} \neq 0$).

Under the same assumptions, suppose we have a line bundle $\mathcal{L}$ over X/C and we wish to lift it to a line bundle $\mathcal{L}'$.

Theorem 1.2. *(Theorem 6.4 of* [Ha]*) Suppose X_0 is nonsingular, and let X be a deformation of X_0 to C with a line bundle $\mathcal{L}$ on it.*

(a) There is an obstruction class in $\mathrm{H}^2(X_0, \mathcal{O}_{X_0} \otimes_k J)$ *which vanishes if and only if $\mathcal{L}$ can be extended to $\mathcal{L}'$ over C'.*

(b) Suppose $\mathrm{H}^0(\mathcal{O}_{X_0}) = k$*. Then the set of such $\mathcal{L}'$ is a torsor under* $\mathrm{H}^1(X_0, \mathcal{O}_{X_0} \otimes_k J)$.

Remark 1.2. 1. It seems one does not even have to assume that X_0 is nonsingular for the above theorem to hold.

2. In [Ha], these cohomology groups are written as $\mathrm{H}^2(X, \mathcal{O}_X \otimes_C J)$ and $\mathrm{H}^1(X, \mathcal{O}_X \otimes_C J)$, but this coincides with what we wrote above. Let us explain why. Recall that higher direct images of pushforward along closed immersions vanish (because j_* is *exact*: this can be checked on stalks). Thus for the inclusion $j : X_0 \to X$ and any sheaf $\mathcal{F}$ on X_0, we have $\mathrm{H}^i(X_0, \mathcal{F}) \cong \mathrm{H}^i(X, j_*\mathcal{F})$. So let us explain why $j_*(\mathcal{O}_{X_0} \otimes_k J) = \mathcal{O}_X \otimes_C J$. To do this a bit more precisely, consider the commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\;j\;} & X \\ \downarrow{\scriptstyle s_0} & & \downarrow{\scriptstyle s} \\ \mathrm{Spec}k & \xrightarrow{\;i\;} & \mathrm{Spec}C \end{array} .$$

Then we think of J as a sheaf over $\mathrm{Spec}C$, and what we wish to prove is that $s^*J = j_*s_0^*i^*J$. As j is a closed immersion, we have that j_* and j^* induce an equivalence of categories between sheaves on X which are supported on X_0 and between sheaves on X; similarly for i_* and i^*. We have that s^*J is supported on X_0: indeed, X_0 is the special fiber of X over $\mathrm{Spec}k \to \mathrm{Spec}C$, and J itself is supported on $\mathrm{Spec}k$, so $(s^*J)_P = \mathcal{O}_{X,P} \otimes_{C_{s(P)}} J_{s(P)}$ is zero for $P \notin X_0$. Thus, we have

$$s^*J = j_*j^*s^*J = j_*(s \circ j)^*J = j_*(i \circ s_0)^* = j_*s_0^*i^*J,$$

as required.

Example 1.2. 1. If X_0/k is affine, any line bundle $\mathcal{L}$ on X can be extended to X' uniquely.

2. If X_0/k is a projective curve, by dimension vanishing H^2 is always 0 for any coherent sheaf, so it can always be deformed from X to X', but usually not uniquely. The vector space $\mathrm{H}^1(X, \mathcal{O}_{X_0}) \cong \mathrm{H}^0(X_0, \Omega^1_{X_0})$ has dimension g over k, so there are g-dimensions in which one can deform.

2. The universal deformation space

Under some conditions there exists a universal deformation space whose points parametrize all deformations of a scheme X_0/k or of lifts of a line bundle on it to a given lift. **(!)** More precisely, there is an affine formal scheme $\mathrm{Spf} R$ over $\mathrm{Spf} W(k)$ which pro-represents deformations Artinian rings A over k, i.e. $\mathrm{Hom}_{W(k)-\mathrm{cont}}(R, A) \cong \{\text{Deformations to } A\}$, whatever the correct notion of deformations is.

Remark 2.1. There also exist notions of versal and miniversal deformation spaces which are slightly more relaxed than pro-representability as above and exist in more general cases, but we will not be concerned with that here.

First we deal with deformations of the scheme X_0/k. Following 3.18 of [Ha] , by a deformation of X_0 to an Artinian ring over k we shall mean a pair (X, i) of a flat scheme X over A together with a closed immersion $i : X_0 \to X$ which induces an isomorphism $X_0 \xrightarrow{\sim} X \times_A k$.

Theorem 2.1. *(Corollary 18.3 of Hartshorne) Let X_0/k be a projective scheme with* $\mathrm{H}^0(X_0, \mathcal{T}_{X_0}) = 0$ *(no infinitesimal automorphisms). Then X_0 has a universal deformation space.*

Example 2.1. 1. The scheme $\mathbb{P}^1_k$ is rigid since $\mathrm{H}^1(X_0, \mathcal{T}_{X_0}) = 0$, so it has a universal deformation space given by $\mathrm{Spf} W(k)$, a point. Note however that it does not satisfy the condition of the theorem.

2. If X is a curve of genus $g \geq 2$ then $\mathcal{T}_{X_0}$ has no global sections because of degree considerations, so it has a universal deformation space. **(!)** Since $\mathrm{H}^2 = 0$ always and $\mathrm{H}^1(X_0, \mathcal{T}_{X_0})$ is $3g - 3$ dimensional, this universal deformation space is $\cong \mathrm{Spf} W(k)[[x_1, ..., x_{3g-3}]]$. **(!)** Another way to think of this is that we are thinking of X_0 as a point in the special fiber of $(\mathcal{M}_g)_{\mathbb{Z}_p}$, and formal completion at that point gives us the universal deformation space.

3. (Ex 18.4.2 in [Ha]) If X is of genus 1, then it has $\mathrm{H}^1(X_0, \mathcal{T}_{X_0}) \neq 0$ and in fact there is no universal deformation space. This is an issue of having too many infinitesimal automorphisms. However if we add to the deformation the data of a choice of a point, this deformation problem is well behaved if we start with a curve with no extra automorphisms (i.e. $j \neq 0, 1728$). In that case there does exist a universal deformation space. It is the formal completion of $\left(\mathbb{A}^1_j\right)_{\mathbb{Z}_p} - \{0, 1728\}$ at the corresponding point of the special fiber.

(!) For line bundles, there is the following deformation problem. Fix X_0/k and a line bundle $\mathcal{L}_0$ on it. Given the category of Artinian algebras over $W(k)$, we consider the deformation problem given by the functor

$$A \mapsto \left\{(X, i, \mathcal{L}) : X/A \text{ flat}, i : X_0 \hookrightarrow X \text{ c.i.}, X_0 \xrightarrow{\sim} X \times_A k, \mathcal{L} \text{ l.b.}, \mathcal{L} \times_A k \cong \mathcal{L}\right\} / \cong .$$

The following theorem seems to be known (relative pro-representability of the local Picard functor), although I have not been able to find a reference for this exact statement: only for variants where we only consider Artinian k-algebras. Also Ravi Vakil in class 18 of his deformation class seems to be mentioning this result is true but does not prove it. It should follow from a standard argument using Schlessinger's criterion.

Theorem 2.2. *Suppose X_0/k is a smooth projective scheme with* $\mathrm{H}^0(X_0, \mathcal{T}_{X_0}) = 0$ *and* $\mathrm{H}^0(X_0, \mathcal{O}_{X_0}) = k$. *Then the functor above has a universal deformation space.*

Note that this deformation space is going to have tangent space of dimension $h^1(X_0, \mathcal{T}_{X_0}) + h^1(X_0, \mathcal{O}_{X_0})$.

Definition. 1. For $\mathbb{P}^1_k$ this universal deformation space should be a point, because both $\mathbb{P}^1_k$ and every line bundle on $\mathbb{P}^1_k$ deform uniquely.
2. If X is a curve of genus $g \geq 2$ this space should exist and it should have dimension $3g - 3 + g = 4g - 3$.

3. From formal Schemes to Varieties

In some situations we have a method from passing from formal schemes to varieties. The key theorem is the following.

Theorem 3.1. *(Theorem 21.2 and Exercise 21.2 of* [Ha]*) Let $\mathcal{X}$ be a formal scheme proper over* $\mathrm{Spf}W(k)$, *and suppose there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}_0 = \mathcal{L} \times_{W(k)} k$ is ample on $X_0 = \mathcal{X} \times_{W(k)} k$. Then there exists a unique scheme X together with an ample line bundle L with $\widehat{X} = \mathcal{X}$ and $\widehat{L} = \mathcal{L}$ when taking completion along the special fiber over $W(k)$.*

Now let X_0 be a smooth projective scheme over k with a line bundle $\mathcal{L}_0$ such that the universal deformation space as in Theorem 2.2 exists. Thus we have a universal deformation space $\mathrm{Spf}R \to \mathrm{Spf}W(k)$, such that for any Artinian $W(k)$-algebra, we have

$$\mathrm{Hom}_{W(k)-\mathrm{cont}}(R, A) \cong \{(X, i, \mathcal{L})\}_A / \cong .$$

We can interpert the $W(k) = \lim W(k)/p^n$-points of R as

$$\begin{aligned}\mathrm{Hom}_{W(k)-\mathrm{cont}}(R, W(k)) &\cong \lim \mathrm{Hom}_{W(k)-\mathrm{cont}}(R, W(k)/p^n) \\ &= \lim \{(X_n, i, \mathcal{L}_n)\} = \{(\mathrm{colim} X_n, i, \lim \mathcal{L}_n)\} / \cong .\end{aligned}$$

Hence, by Proposition 2.1 [Ha] each $W(k)$-point gives us a formal scheme $\mathcal{X} = \mathrm{colim} X_n$ which is proper and flat over $\mathrm{Spf}W(k)$ and whose special fiber is X_0, together with a line bundle $\mathcal{L} = \lim \mathcal{L}_n$. Applying the above theorem to $\mathcal{X}$, we obtain the following theorem.

Theorem 3.2. *Suppose X_0 is a nonsingular projective scheme with a line bundle $\mathcal{L}_0$ such that the universal deformation space for deformations $(X, i, \mathcal{L})$ exists. Then the $W(k)$- points of the universal deformation parametrize the liftings of X_0 to a flat projective variety over $W(k)$ together with a lift of its ample line bundle.*

References

[Ha] Deformation theory, Hartshorne.