ATTACHING GALOIS REPRESENTATIONS TO MODULAR FORMS

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Abstract. These are notes for an introductory talk which explains Deligne's recipe for attaching a Galois representation to a newform of weight $k \geq 2$.

References: [De71], [Sch90] and notes for the talk "Attaching l-adic representations to elliptic modular forms" by Jay Pottharst.

1. Introduction

Let f be a cuspidal normalized newform of weight k for $k \geq 2$, level $N \geq 5$ and Fourier coefficients a_n. Let l be a prime.

Theorem 1.1. *(Deligne) There exists a continuous semisimple representation* $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ *which is unramified for* $p \nmid lN$*, such that* $\mathrm{Tr}(\rho_f(\mathrm{Frob}_p^{-1})) = a_p$.

In fact, Deligne shows exactly where one can find this representation. Fix once and for all a level $\Gamma = \Gamma(N)$ and let $\mathcal{E} = \mathcal{E}(\Gamma)$ be the universal elliptic curve lying over $Y = Y(\Gamma)$, with the structure morphism $\pi : \mathcal{E} \to Y$. Then ρ_f is a subquotient of the etale cohomology group

$$\mathrm{H}^1_{\mathrm{par}}\left(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2} R^1_{\text{ét}} \pi_* \mathbb{Q}_l\right).$$

Here, $\mathrm{H}^i_{\mathrm{par}} := \mathrm{Im}\left(\mathrm{H}^i_c \to \mathrm{H}^i_{\text{ét}}\right)$ is a surrogate for $\mathrm{H}^1_{\text{ét}}$ which appears because Y has cusps. The purpose of this talk is to explain what this means and to give a sketch of the construction.

We can describe this subquotient as follows. Let $\mathbb{T}(N)$ be the Hecke algebra generated over $\mathbb{Q}$ by the Hecke operators T_p away from N. Using the language of correspondences, this Hecke algebra acts on $\mathrm{H}^1_{\mathrm{par}}\left(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2} R^1_{\text{ét}} \pi_* \mathbb{Q}_l\right)$. (and on any other cohomology group we will see). Let K_f be the number field generated by the coefficients of f. We have a homomorphism $\mathbb{T}(N) \to K_f$ given by sending T_p to a_p. Choose a place λ of $K_{f,\lambda}$ lying over l, and form the λ-adic completion $K_{f,\lambda}$. Then it will turn out that

$$V_f = \mathrm{H}^1_{\mathrm{par}}\left(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2} R^1_{\text{ét}} \pi_* \mathbb{Q}_l\right) \otimes_{\mathbb{T}(N)} K_{f,\lambda}$$

defines the 2-dimensional representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\lambda}) \subset \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ associated to f.

Example 1.1. Take $k = 2$. Then $\mathrm{Sym}^{k-2} R^1_{\text{ét}} \pi_* \mathbb{Q}_l = \mathbb{Q}_l$, and

$$\mathrm{H}^1_{\mathrm{par}}\left(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_l\right) \cong \mathrm{H}^1_{\text{ét}}\left(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l\right) \cong$$

$$\cong \mathrm{H}^1_{\text{ét}}\left(J\left(X_{\overline{\mathbb{Q}}}\right), \mathbb{Q}_l\right) \cong \left(\left(\lim_{\infty \leftarrow l} \mathrm{Jac}\left(X_{\overline{\mathbb{Q}}}\right)[l^n]\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\right)^{\vee},$$

so that V_f can be found in the dual of a Tate module of the Jacobian of $X_{\overline{\mathbb{Q}}}$.[1]

Example 1.2. Take $k = 3$. The Leray spectral sequence implies that

$$\mathrm{H}^1_{\text{ét}}\left(Y_{\overline{\mathbb{Q}}}, R^1_{\text{ét}}\pi_*\mathbb{Q}_l\right) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}\left(\mathcal{E}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l\right),$$

and so $\mathrm{H}^1_{\text{par}}\left(Y_{\overline{\mathbb{Q}}}, R^1_{\text{ét}}\pi_*\mathbb{Q}_l\right)$, and hence V_f, is a subquotient of $\mathrm{H}^2_{\text{ét}}\left(\mathcal{E}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l\right)$, some 6-dimensional Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Remark 1.1. For $k \geq 4$, there is a description similar to the latter example of $\mathrm{H}^1_{\text{par}}\left(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2} R^1_{\text{ét}}\pi_*\mathbb{Q}_l\right)$ as a subquotient of an étale cohomology group of a (resolution of) a symmetric power of $\mathcal{E}_{\overline{\mathbb{Q}}}$. See section 5 of [De71] or the main result of [Sch90] for a more precise result.

What needs to be explained is why ρ_f is 2-dimensional, why it is unramified for $p \nmid lN$ and why $\mathrm{Tr}(\rho_f(\mathrm{Frob}_p^{-1})) = a_p$.

2. Modular forms

As a first step, let us explain how modular forms can be interperted as sections of coherent sheaves. Let $\mathcal{B} = \{(z_1, z_2) : \mathrm{Im}(z_1/z_2)\}$ be the set of oriented bases of $\mathbb{C}$. We may think of it as an analytic space over $\mathbb{C}$. It is endowed with a left action of $\mathbb{C}^\times$ given by $\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$ and a right action of $\mathrm{SL}_2(\mathbb{Z})$ given by $(z_1, z_2)\begin{pmatrix} a & c \\ b & d \end{pmatrix} = (az_1 + bz_2, cz_1 + dz_2)$. For $\Gamma(N) :=$ $\mathrm{Ker}\left(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right)$, we can form the quotients $\mathbb{H} = \mathbb{C}^\times \backslash \mathcal{B}$ and $\mathcal{L}(N) = \mathcal{B}/\Gamma(N)$ are the upper half plane and the space of lattices with level N structures. The double quotient $Y(N) = \mathbb{C}^\times \backslash \mathcal{B}/\Gamma(N)$ is also called a modular curve.

These spaces can all be thought of as moduli spaces of elliptic curves with extra structure:

- The modular curve $Y(N)$ parametrizes pairs $\left(E, E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2\right)$ of elliptic curves E over $\mathbb{C}$ and a trivialization for their N torsion;
- The space $\mathcal{L}(N)$ parametrizes triples $\left(E, E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2, \omega_E \xrightarrow{\sim} \mathbb{C}\right)$, where $\omega_E = \mathrm{Lie}_{\mathbb{C}}(E)^\vee$ is the cotangent space;
- The space $\mathbb{H}$ parametrizes pairs $\left(E, \mathrm{H}_1(E, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^2\right)$;
- Finally, the space $\mathcal{B}$ parametrizes triples $\left(E, \omega_E \xrightarrow{\sim} \mathbb{C}, \mathrm{H}_1(E, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^2\right)$.

They sit in a nice diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Gamma(N)} & \mathcal{L}(N) \\ \downarrow{\scriptstyle \mathbb{C}^\times} & & \downarrow{\scriptstyle \mathbb{C}^\times} \\ \mathbb{H} & \xrightarrow{\Gamma(N)} & Y(N) \end{array}.$$

The classical definition of holomorphic modular forms of weight k and level N is of functions on the upper half plane $\mathbb{H}$ satisfying the identity $(f\gamma)(\tau) = j(\gamma, \tau)^k f(\tau)$ for γ in $\Gamma(N)$. We shall write this as $\mathrm{H}^0(\mathbb{H}, \mathcal{O}_{\mathbb{H}})^{\Gamma(N) = j^k}$. On the other hand, let χ_{-k} be the character of $\mathbb{C}^\times$

[1] In some sources, the normalization has it so that V_f is defined to be the dual of our V_f, so that it appears in the Tate module of a Jacobian, instead of its dual. This has to do with whether or not we are requiring that $\mathrm{Tr}(\rho_f(\mathrm{Frob}_p^{-1})) = a_p$ or $\mathrm{Tr}(\rho_f(\mathrm{Frob}_p)) = a_p$

which sends λ to λ^{-k}. The diagram and standard properties of covering maps then show that

$$\{\text{Modular functions of weight } k, \text{ level } \Gamma(N)\} = \mathrm{H}^0\left(\mathbb{H}, \mathcal{O}_{\mathbb{H}}\right)^{\Gamma(N)=j^k} \xrightarrow{\sim}$$

$$\mathrm{H}^0\left(\mathcal{B}, \mathcal{O}_{\mathcal{B}}\right)^{\Gamma(N)=1, \mathbb{C}^\times=\chi_{-k}} \xrightarrow{\sim} \mathrm{H}^0\left(\mathcal{L}\left(N\right), \mathcal{O}_{\mathcal{L}(N)}\right)^{\mathbb{C}^\times=\chi_{-k}} \xrightarrow{\sim} \mathrm{H}^0\left(Y\left(N\right), \omega^{\otimes k}\right),$$

with ω being the invertible sheaf corresponding to the geometric line bundle $\mathcal{L}(N) \to Y(N)$.

The space $Y(N)$ can be compactified to a space $X(N)$ by adding finitely many pointes called cusps, and ω extends to a line bundle on it. There is the Kodaira-Spencer isomorphism

$$\Omega^1_{X(N)} \xrightarrow{\sim} \omega^{\otimes 2}\left(-\text{cusps}\right).$$

This shows that cusp forms of weight k (these modular forms which vanish at the cusps) are given by

$$S_k(\Gamma(N)) = \mathrm{H}^0\left(X\left(N\right), \Omega^1_{X(N)} \otimes \omega^{\otimes k-2}\right).$$

This interpertation of modular forms has two advantages. First, it gives a description of modular forms in terms of coherent cohomology; second, it involves the spaces $Y(N)$ and $X(N)$, rather than the much larger and non-algebraizable[2] space $\mathbb{H}$.

3. Hodge theory and the Eichler-Shimura isomorphism

To obtain Galois representations, we will ultimately need to relate these coherent cohomology groups to étale cohomology groups. A first step is to relate coherent cohomology to singular cohomology, which can be done by using Hodge theory.

Let $\mathcal{E}$ the universal elliptic curve lying over $Y(N)$. If $\pi : \mathcal{E} \to Y(N)$ is the structure map, then $\omega = \pi_* \Omega^1_{\mathcal{E}/Y(N)}$. For each $x \in Y(N)$ taking the fiber gives rise to an elliptic curve $\mathcal{E}_x$ over $\mathbb{C}$ and a de Rham exact sequence of sheaves over $\mathcal{E}_x$:

$$0 \to \underline{\mathbb{C}} \to \mathcal{O}_{\mathcal{E}_x} \xrightarrow{d} \Omega^1_{\mathcal{E}_x} \to 0.$$

Taking cohomology, we obtain a connecting homomorphism

$$\mathrm{H}^0\left(x, \omega_x\right) = \mathrm{H}^0\left(\mathcal{E}_x, \Omega^1_{\mathcal{E}_x}\right) \to \mathrm{H}^1\left(\mathcal{E}_x, \underline{\mathbb{C}}\right) = \left(R^1\pi_*\underline{\mathbb{C}}\right)_x.$$

Letting x vary, we obtain a map $\omega \to R^1\pi_*\underline{\mathbb{C}}$ of sheaves on $Y(N)$. It is this map which enables us to relate coherent cohomology and singular cohomology. It induces a map $\omega^{k-2} \to \mathrm{Sym}^{k-2}\omega \to \mathrm{Sym}^{k-2}\left(R^1\pi_*\underline{\mathbb{C}}\right)$, and taking parabolic cohomology yields a map

$$\mathrm{H}^1_{\text{par}}\left(Y(N), \omega^{k-2}\right) \to \mathrm{H}^1_{\text{par}}\left(Y(N), \mathrm{Sym}^{k-2}\left(R^1\pi_*\underline{\mathbb{C}}\right)\right).$$

To relate this to modular forms, we note the de Rham exact sequence $0 \to \underline{\mathbb{C}} \to \mathcal{O}_{Y(N)} \xrightarrow{d} \Omega^1_{Y(N)} \to 0$ yields a map

$$\mathrm{H}^0_{\text{par}}\left(Y(N), \Omega^1 \otimes \omega^{k-2}\right) \to \mathrm{H}^1_{\text{par}}\left(Y(N), \omega^k\right),$$

and $Y(N) \to X(N)$ induces $S_k\left(\Gamma(N)\right) = \mathrm{H}^0\left(X(N), \Omega^1 \otimes \omega^{k-2}\right) \to \mathrm{H}^0_{\text{par}}\left(Y(N), \Omega^1 \otimes \omega^{k-2}\right)$. Composing all of these together, we obtain the Eichler-Shimura map

$$\mathrm{ES} : S_k\left(\Gamma(N)\right) \to \mathrm{H}^1_{\text{par}}\left(Y(N), \mathrm{Sym}^{k-2}\left(R^1\pi_*\underline{\mathbb{C}}\right)\right).$$

[2]It is often very useful that by this interpertation modular forms can be viewed as sections not only in the complex but also in the algebraic category, but we will have no use of that here.

The $\mathbb{Q}$-algebra $\mathbb{T}(N)$ acts on both of $S_k(\Gamma(N))$ and $\mathrm{H}^1_{\mathrm{par}}\left(Y(N), \mathrm{Sym}^{k-2}\left(R^1\pi_*\underline{\mathbb{C}}\right)\right)$ via correspondences. It turns out that ES is $\mathbb{T}(N)$-equivariant.

One then has the following fundamental theorem, whose proof we omit.

Theorem 3.1. *(The Eichler Shimura isomorphism) The map*

$$\mathrm{ES} \oplus \overline{\mathrm{ES}} : S_k(\Gamma(N)) \oplus \overline{S_k(\Gamma(N))} \to \mathrm{H}^1_{\mathrm{par}}\left(Y(N), \mathrm{Sym}^{k-2}\left(R^1\pi_*\underline{\mathbb{C}}\right)\right)$$

is a $\mathbb{T}(N)$-equivariant isomorphism.

4. Étale cohomology and the Eichler-Shimura relations

We now need to replace the complex singular cohomology group

$$W_{\mathbb{C}} = \mathrm{H}^1_{\mathrm{par}}\left(Y(N), \mathrm{Sym}^{k-2}\left(R^1\pi_*\mathbb{C}\right)\right)$$

by an étale cohomology group. To do this, we consider its $\mathbb{Q}$-form given by

$$W_{\mathbb{Q}} = \mathrm{H}^1_{\mathrm{par}}\left(Y(N), \mathrm{Sym}^{k-2}\left(R^1\pi_*\mathbb{Q}\right)\right),$$

i.e. $W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \cong W_{\mathbb{C}} \cong S_k(\Gamma(N)) \oplus \overline{S_k(\Gamma(N))}$. On the other hand, this is also a $\mathbb{Q}_l$-form for an etale cohomology group. Indeed, the Artin comparison theorem gives

$$W_{l,\overline{\mathbb{Q}}} := \mathrm{H}^1_{\mathrm{par}}\left(Y(N)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2}\left(R^1_{\text{ét}}\pi_*\mathbb{Q}_l\right)\right)$$

$$\xrightarrow{\sim} \mathrm{H}^1_{\mathrm{par}}\left(Y(N), \mathrm{Sym}^{k-2}\left(R^1\pi_*\mathbb{Q}\right)\right) \otimes_{\mathbb{Q}} \mathbb{Q}_l = W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_l.$$

Setting $V_f = \mathrm{H}^1_{\mathrm{par}}\left(Y_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2}\left(R^1_{\text{ét}}\pi_*\mathbb{Q}_l\right)\right) \otimes_{\mathbb{T}(N)} K_{f,\lambda}$ for λ a place of K_f lying over l, we may now compute that

$$\begin{aligned}\dim_{K_{f,\lambda}} V_f &= \mathrm{rank}_{K_f\otimes\mathbb{Q}_l} W_{l,\overline{\mathbb{Q}}} \otimes_{\mathbb{T}(N)} K_f \\ &= \dim_{K_f} W_{l,\overline{\mathbb{Q}}} \otimes_{\mathbb{T}(N)} K_f \\ &= \dim_{K_f\otimes_{\mathbb{Q}}\mathbb{C}}\left(\left(S_k(\Gamma(N)) \oplus \overline{S_k(\Gamma(N))}\right) \otimes_{\mathbb{T}(N)} K_f\right) = 2\dim_{K_f\otimes_{\mathbb{Q}}\mathbb{C}} S_k(\Gamma(N)) \otimes_{\mathbb{T}(N)} K_f.\end{aligned}$$

Since f is a newform, multiplicity 1 implies that $S_k(\Gamma(N)) \otimes_{\mathbb{T}(N)} K_f \cong K_f \otimes_{\mathbb{Q}} \mathbb{C}$, so that $\dim_{K_{f,\lambda}} V_f = 2$.

To study this representation, we shall relate $W_{l,\overline{\mathbb{Q}}}$ as a representation of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$, to $W_{l,\overline{\mathbb{F}}_p} := \mathrm{H}^1_{\mathrm{par}}\left(Y_{\overline{\mathbb{F}}_p}, \mathrm{Sym}^{k-2}R^1_{\text{ét}}\pi_*\mathbb{Q}_l\right)$ as a representation of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Now Y has a smooth model over $\mathrm{Spec}\mathbb{Z}[\zeta_n][1/N]$, and if we let $t : Y \to \mathrm{Spec}\mathbb{Z}[\zeta_n][1/N]$ be the structure morphism, it follows from the smooth base change theorem that $W_{l,\overline{\mathbb{Q}}}$ and $W_{l,\overline{\mathbb{F}}_p}$ are specializations of $W_l = R^1_{\text{ét}}t_*\mathrm{Sym}^{k-2}R^1_{\text{ét}}\pi_*\mathbb{Q}_l$ at geometric points. It then follows from functoriality that we have a Hecke-equivariant identification

$$\left(W_{l,\overline{\mathbb{F}}_p}, \mathrm{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right)\text{-action}\right) \cong \left(W_{l,\overline{\mathbb{Q}}}, \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)\text{-action restricted to a decomposition group at } p\right).$$

In particular, we see that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ restricted to a decomposition group factors through $\mathrm{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right)$, and so is unramified.

Finally, we check that $\mathrm{Tr}(\rho_f(\mathrm{Frob}_p^{-1})) = a_p$, i.e. that $\mathrm{Tr}(\mathrm{Frob}_p^{-1}) = a_p$ on $W_{l,\overline{\mathbb{Q}}} \otimes_{\mathbb{T}(N)} K_f$. It is enough to study $W_{l,\overline{\mathbb{F}}_p}$ with its associated $\mathrm{Gal}\left(\overline{\mathbb{F}}_p/\mathbb{F}_p\right)$ and $\mathbb{T}(N)$ actions, and show that $\mathrm{Tr}\left(\mathrm{Frob}_p^{-1}\right) = T_p$, because tensoring with K_f equates T_p and a_p.

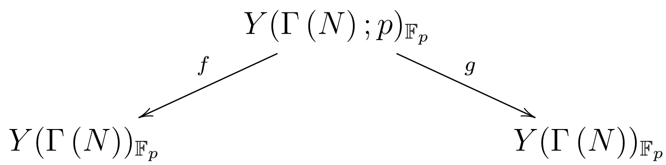
Let $F : W_{l,\overline{\mathbb{F}}_p} \to W_{l,\overline{\mathbb{F}}_p}$ be the map induced by geometric Frobenius morphism Frob. Explictly, F is given by the action on $W_{l,\overline{\mathbb{F}}_p}$ induced by the map Frob on $Y_{\overline{\mathbb{F}}_p}$ sending $(E, \text{level structure})$ to $\left(E^{(p)}, \text{level structure}\right)$. We also have the map V induced from Frobenius in the reverse direction in cohomology. There is a general étale cohomology property which says that in characteristic p, the action of Frob_p^{-1} is equal to F. This implies that

$$\mathrm{Tr}\left(\mathrm{Frob}_p^{-1} | W_{l,\overline{\mathbb{F}}_p}\right) = F + V.$$

It now remains to establish the following theorem.

Theorem 4.1. *(The Eichler-Shimura relation) We have* $T_p = F + V$ *on* $W_{l,\overline{\mathbb{F}}_p}$.

Proof. We can explicitly compute the correspondence T_p. We have the curve $Y(\Gamma(N);p)_{\mathbb{F}_p}$ over $\mathrm{Spec}\mathbb{F}_p$ whose points parametrize triples $\left(E, \alpha : E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2, C \hookrightarrow E[p]\right)$ where C is a subgroup scheme of order p. The correspondence T_p is given by



where $f(E, *, C) = (E, *)$ and $g(E, *, C) = (E/C, *)$.

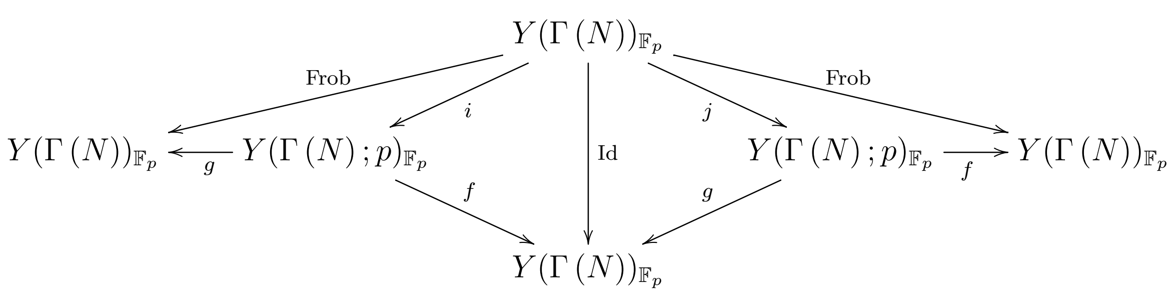
On the other hand, we can define two maps $i, j : Y(\Gamma(N))_{\mathbb{F}_p} \to Y(\Gamma(N);p)_{\mathbb{F}_p}$ in the opposite direction, given by

$$i(E, *) = \left(E, *, \mathrm{Ker}\left(\mathrm{Frob} : E \to E^{(p)}\right)\right)$$

and

$$j(E, *) = \left(E^{(p)}, *, \mathrm{Ker}\left(\widehat{\mathrm{Frob}} : E^{(p)} \to E\right)\right).$$

It is then straightforward to check that $g \circ i = f \circ j = \mathrm{Frob}$ and $f \circ i = g \circ j = \mathrm{Id}$, so that we have the following diagram:



Now $f \circ i = g \circ j = \mathrm{Id}$ means that i and j are closed immersions. On the other hand, for ordinary curves over $\mathbb{F}_p$, every subgroup scheme of order p can be written as either μ_p or $\mathbb{Z}/p\mathbb{Z}$. In particular, a point $(E, *, C) \in Y(\Gamma(N);p)_{\mathbb{F}_p}^{\mathrm{ord}}$ is in the image of i if $C = \mu_p$ and in the image of j if $C = \mathbb{Z}/p\mathbb{Z}$, and we get that

$$(i, j) : Y(\Gamma(N))_{\mathbb{F}_p}^{\mathrm{ord}} \amalg Y(\Gamma(N))_{\mathbb{F}_p}^{\mathrm{ord}} \xrightarrow{\sim} Y(\Gamma(N);p)_{\mathbb{F}_p}^{\mathrm{ord}}$$

is an isomorphism. We can then reinterpert the correspondence T_p on the ordinary locus by replacing $Y(\Gamma(N);p)^{\mathrm{ord}}_{\mathbb{F}_p}$ with $Y(\Gamma(N))^{\mathrm{ord}}_{\mathbb{F}_p} \amalg Y(\Gamma(N))^{\mathrm{ord}}_{\mathbb{F}_p}$, and we see that T_p is the sum of the two correspondences $(\mathrm{Frob}, 1) = F$ and $(1, \mathrm{Frob}) = V$ on the ordinary locus. But the ordinary locus is dense, so $T_p = F + V$. □

References

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