ADIC SPACES

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ABSTRACT. These are notes for a talk which gives a very brief introduction to the theory of adic spaces. We present the concept with an emphasis on examples.

1. Huber Rings

The coordinate rings of affinoid adic rings are Huber rings. We begin by giving the relevant definitions with some examples.

Definition 1.1. Let A be a topological ring.
1. A is called *adic* if there exists an ideal I in A such that $\{I^n\}_{n\geq 1}$ give a basis of neighborhoods at 0. Such an ideal is called an ideal of definition.
2. A is called *Huber* if there exists an open subring $A_0 \subset A$ which is adic with a finitely generated ideal of definition. (The ring A_0 is not unique and is not part of the data of a Huber ring).
3. A is called *Tate* if it is Huber with a topologically nilpotent unit.

In the following every ring is endowed with its natural topology.

Example 1.1. 1. The ring $\mathbb{Z}_p$ is adic with $I = (p)$. The rings $\mathbb{Z}_p \langle T \rangle$ and $\mathbb{Z}_p[[T]]$ are adic with $I = (p, T)$.
2. Any ring endowed with the discrete topology is adic with $I = 0$.
3. The ring $\mathbb{Q}_p$ is Huber (with $A_0 = \mathbb{Z}_p$), but not adic. In fact, it is Tate with the topological unit p. Similar remarks hold for $\mathbb{Q}_p \langle T \rangle, \mathbb{Q}_p[[T]]$.
4. The ring $\mathbb{Z}_p \langle T \rangle$ is Huber (with $A_0 = \mathbb{Z}_p \langle T \rangle$), but not Tate.
5. The ring $\mathbb{Z}_p[[T_1, ...T_n, ...]]$ with $I = (T_1, ..., T_n, ...)$ is adic but not Huber, because I is not finitely generated.

Let S be a subset of A. We say S is bounded if for any open neighborhood U of 0 there exists an open neighborhood V of 0 such that $S \cdot V \subset U$. We denote by A° the set of *power-bounded* elements of A. That is, these elements of A whose set of powers form a bounded set.

Suppose that A is Huber. Then it is not too hard to check that A° is the union of all rings of definition. In particular, it is open, and moreover it is integrally closed. If A° is bounded, it is also a ring of definition (and hence the maximal one). In that case we say A is *uniform*.

Example 1.2. 1. The Huber ring $A = \mathbb{Q}_p \langle T_1, ..., T_n \rangle$ is uniform, because $A^\circ = \mathbb{Z}_p \langle T_1, ..., T_n \rangle$.
2. The Huber ring $A = \mathbb{Q}_p \langle T \rangle / T^2$ (with $A_0 = \mathbb{Z}_p \oplus T\mathbb{Q}_p \langle T \rangle$) is not uniform. All the elements of the form $p^n T$, $n \in \mathbb{Z}$ are in A°, because they are nilpotent, but their p-adic valuations are unbounded.

Definition 1.2. A Huber pair is a pair (A, A^+) where A is a Huber ring and $A^+ \subset A^\circ$ is an open and integrally closed subring.

Example 1.3. 1. (A, A°) is always a Huber pair.
2. If A is adic with a finitely generated ideal, (A, A) is a Huber pair.
3. For $A = \mathbb{Q}_p \langle T \rangle$ one may choose $A^+ = A^\circ$, or the slightly smaller subring

$$A^{++} = \left\{ \sum_{n=0}^{\infty} a_n T^n \in A^+ : |a_n| < 1 \text{ for all } n \geq 1 \right\}.$$

2. Valuations and the adic specturm

A *valuation* of a topological ring A is a multiplicative map $|\cdot| : A \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group, which satisfies $|0| = 0$, $|1| = 1$ and $|a+b| \leq \max\{|a|, |b|\}$ for $a, b \in A$. (Really, the name seminorm might be more fitting). It is called *continuous* if the inverse image of each tail $\{a : a < \gamma\}$ in $\Gamma \cup \{0\}$ is open. There is an obvious notion of equivalence of valuations.

Given a Huber pair (A, A^+), we let

$$\mathrm{Spa}(A, A^+) = \left\{ \text{continuous valuations } |\cdot| : |f| \leq 1 \text{ for any } f \in A^+ \right\} / \text{equivalence}.$$

The set $\mathrm{Spa}(A, A^+)$ has the structure of a topological space, where the topology is generated by the open subsets

$$\left\{ |\cdot| \in \mathrm{Spa}(A, A^+) : |f| \leq |g| \neq 0 \right\}$$

for $f, g \in A$. In particular (taking $f = 1$ and reversing the roles of f and g), each set of the form $\{|f| \neq 0\}$ is open. It is customary to sometimes write x for a valuation $|\cdot|$ and $|f(x)|$ for $|f|$.

In general it seems this space can be very complicated, but we can give a few examples.

Example 2.1. 1. Let $(A, A^+) = (\mathbb{Q}_p, \mathbb{Z}_p)$. Then $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ consists only of one point, which is (up to equivalence) the p-adic valuation $|\cdot|_p : \mathbb{Q}_p \to \mathbb{R}_{>0} \cup \{0\}$. Indeed, any valuation has to be trivial on $\mathbb{Z}_p^\times$, and then everything is determined by $|p|$. Since p is a topologically nilpotent unit, one must have $0 < |p| < 1$, and every such choice gives a valuation equivalent to $|\cdot|_p$.
2. Let $(A, A^+) = (\mathbb{Z}_p, \mathbb{Z}_p)$. Then by similar considerations to the above $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ consists of two points: the p-adic valuation $\eta = |\cdot|_p$, and the composition s of the reduction map $\mathbb{Z}_p \to \mathbb{F}_p$ with the trivial valuation on $\mathbb{F}_p$. (The difference from the previous example is that p is no longer a unit so one is allowed to send it to 0). The point η is open: it is the subset $\{|p| \neq 0\}$. On the other hand s is not open. This makes $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ homeomorphic to $\mathrm{Spec}\mathbb{Z}_p$.
3. Let $(A, A^+) = (\mathbb{Z}, \mathbb{Z})$. Then for each prime p we get two points: the point obtained from the composition of $\mathbb{Z} \to \mathbb{F}_p$ with the trivial valuation, and the p-adic topology. We also have trivial norm which sends all nonzero integers to 1. (It might seem like the archimedean norm is missing, but it's not bounded by 1 on $\mathbb{Z}$).

4. Let $(A, A^+) = \left(\mathbb{C}_p \langle T \rangle, \mathcal{O}_{\mathbb{C}_p} \langle T \rangle\right)$. Then one thinks of $\mathbb{D} = \mathrm{Spa}(\mathbb{C}_p \langle T \rangle, \mathcal{O}_{\mathbb{C}_p} \langle T \rangle)$ as the closed adic unit disc. It has several kinds of points, but for sake of brevity we discuss only three classes of them.

(i) There are classical points: each $x \in \mathcal{O}_{\mathbb{C}_p}$ gives a valuation via $\mathbb{C}_p \langle T \rangle \to \mathbb{C}_p \langle T \rangle / (T - x) \cong \mathbb{C}_p \xrightarrow{|\cdot|} \mathbb{R}_{>0} \cup \{0\}$.

(ii) For each disc $D = D(x, r) \subset \mathbb{D}$, there is a point $x_D : \mathbb{C}_p \langle T \rangle \to \mathbb{R}_{>0} \cup \{0\}$ given by

$$|f(x_D)| := \sup_{x \in D} |f(x)|.$$

(iii) For each $D = D(x, r)$ with $r < 1$, there are two additional points $x_D^{\pm}$. The define the point x_D^-, let $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$ where $a < \gamma < 1$ for any $a < 1$ in $\mathbb{R}_{>0}$. Then $x_D^- : \mathbb{C}_p \langle T \rangle \to \Gamma \cup \{0\}$ is given by the formula

$$|f(x_D)| = \sup_{x \in D} |a_n| r^n \gamma^n.$$

The point x_D^+ is defined in a similar way, where we specify instead that $1 < \gamma < a$ for any $1 < a$ in $\mathbb{R}_{>0}$.

For $D = \mathbb{D}$ we also get a such point $x_{\mathbb{D}}^-$, but not a point $x_{\mathbb{D}}^+$, because the latter is not bounded on 1 on $\mathcal{O}_{\mathbb{C}_p} \langle T \rangle$. It can be shown that if A^{++} is taken as in Example 1.3.3 then $\mathrm{Spa}(A, A^{++}) = \mathbb{D} \cup \{x_{\mathbb{D}}^+\}$.

In the above, the points of type (i) and (iii) are closed, but points of (ii) are not: the points $x_D^{\pm}$ are in their closure. Later, we will also see that $\mathbb{D}$ is connected.

As a topological space, $\mathrm{Spa}(A, A^+)$ has the following properties.

Theorem 2.1. *1. The space* $\mathrm{Spa}(A, A^+)$ *is spectral; i.e., there exists a ring R and a homeomorphism* $\mathrm{Spa}(A, A^+) \cong \mathrm{Spec} R$. *In particular, it is quasi compact, and every irreducible set has a generic point.*

2. A basis of open sets is given by that rational subsets. These are sets of the form

$$U(T/s) = \{x \in \mathrm{Spa}(A, A^+) \text{ such that for any } t \in T, |t(x)| \leq |s(x)| \neq 0\},$$

where $s \in A$ and $T \neq \emptyset$ is a finite subset of A such that $T \cdot A \subset A$ is an open ideal.

Remark 2.1. 1. In part 1, the analogy with schemes should not be taken too strongly. For example, with schemes one is sometimes accustomed to thinking of points as being generic points of irreducible subschemes. But even for $\mathbb{D}$ it is not true that a subdisc is a generization of the points lying inside it.

2. If A is a Tate algebra, the condition that $T \cdot A \subset A$ is open is saying that $T = (1)$, so this is reminiscient of the condition one sees in rigid geometry.

3. Adic spaces

Let $s \in A$ and T as in Theorem 2.1, and write $T = \{t_1, ..., t_n\}$. We let $A \langle T/s \rangle = A\left(\frac{t_1}{s}, ..., \frac{t_n}{s}\right)^{\wedge}$ and $A \langle T/s \rangle^+$ the completion of the integral closure of $A^+\left(\frac{t_1}{s}, ..., \frac{t_n}{s}\right)$. One can prove that $\left(A \langle T/s \rangle, A \langle T/s \rangle^+\right)$ form a Huber pair.

Proposition 3.1. *The natural map* $\mathrm{Spa}(A\langle T/s\rangle, A\langle T/s\rangle^+) \to \mathrm{Spa}(A, A^+)$ *is an open embedding with image* $U(T/s)$*. Moreover, rational subsets of* $\mathrm{Spa}(A\langle T/s\rangle, A\langle T/s\rangle^+)$ *correspond to rational subsets of* $\mathrm{Spa}(A, A^+)$ *contained in* $U(T/s)$*.*

Given a Huber pair (A, A^+), we may define a structures of presheafs $\mathcal{O}_X$ and $\mathcal{O}_X^+$ of complete topological rings on $X = \mathrm{Spa}(A, A^+)$ as follows. For a rational subset $U(T/s)$, we let

$$\mathcal{O}_X(U(T/s)) = A\langle T/s\rangle$$

and

$$\mathcal{O}_X^+(U(T/s)) = A\langle T/s\rangle^+ .$$

The proposition insures us that there are transition maps whenever one rational set is contained in another. In fact, the sets $\mathrm{Spa}(A\langle T/s\rangle, A\langle T/s\rangle^+)$ satisfy a universal property, so these maps are canonical. This allows one to extend $\mathcal{O}_X$ and $\mathcal{O}_X^+$ to presheaves on X.

Definition 3.1. A Huber pair (A, A^+) is called *sheafy* if $\mathcal{O}_X$ is a sheaf on $\mathrm{Spa}(A, A^+)$.

If (A, A^+) is sheafy, it turns out the stalks of $\mathcal{O}_X$ are local rings. The reason is that for every $x \in \mathrm{Spa}(A, A^+)$, the valuation v_X corresponding to x glues over $\mathcal{O}_X(U)$ for rational subsets U containing x. By passing to the limit one obtains a valuation on $\mathcal{O}_{X,x}$, and one can show it is a local ring with maximal ideal being the kernel of the valuation v_x. Note that $\mathcal{O}_X^+$ is equivalent to the data of the v_x, because for each rational subset U we have

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leq 1 \text{ for } x \in U\} .$$

Let $\mathcal{V}$ be the category consisting of topological spaces X with a sheaf of complete topological rings whose stalks $\mathcal{O}_{X,x}$ are local, together with a valuation v_x on $\mathcal{O}_{X,x}$. Thus a sheafy Huber pair (A, A^+) gives rise to an object $\mathrm{Spa}(A, A^+)$ of $\mathcal{V}$.

Definition 3.2. An *adic space* is an object of $\mathcal{V}$ which is locally isomorphic to $\mathrm{Spa}(A, A^+)$ for a sheafy Huber pair (A, A^+). An *affionoid adic space* is an object isomorphic to $\mathrm{Spa}(A, A^+)$ for some (A, A^+).

There remains the question of which Huber pairs are sheafy. Fortunately, many cases are taken care of by the following theorem.

Theorem 3.1. *Let* (A, A^+) *be a Huber pair. Then* (A, A^+) *is sheafy in each of the following situations.*

1. A has the discrete topology.

2. A is Tate and $A\langle X_1, ..., X_n\rangle$ *is noetherian for all* $n \geq 1$*.*

3. A is finitely generated over a noetherian ring of definition.

4. Stable uniformity: A is Tate and for each rational subset U the Tate ring $\mathcal{O}_X(U)$ *is uniform.*

Example 3.1. 1. Condition 1 allows us to view schemes as adic spaces. In other words, there is a fully faithgl functor which extends $\mathrm{Spec} A \mapsto \mathrm{Spa}(A, A)$.

2. Condition 2 allows us to view rigid spaces as adic spaces. Here if A is a Tate algebra then $\mathrm{Sp}(A)$ is mapped to $\mathrm{Sp}(A)^{\mathrm{ad}} = \mathrm{Spa}(A, A^\circ)$, which extends to a functor that preserves open

immersions. Given a rigid space X and a family of admissible open subsets $\{U_i\}_{i\in I}$ of X, the family form an admissible cover of X if and only if $\{U_i^{\mathrm{ad}}\}_{i\in I}$ cover X^{ad} as topological spaces.

For instance, in Example 2.1.4 we have $\mathbb{D} = \mathrm{Sp}\left(\mathbb{C}_p\left\langle T\right\rangle\right)^{\mathrm{ad}}$. Recall the well known example of a nonadmissible covering of $\mathrm{Sp}\left(\mathbb{C}_p\left\langle T\right\rangle\right)$. One takes

$$U = \{x \in \mathrm{Sp}\left(\mathbb{C}_p\left\langle T\right\rangle\right) : |x| < 1\}, V = \{x \in \mathrm{Sp}\left(\mathbb{C}_p\left\langle T\right\rangle\right) : |x| = 1\}.$$

Then U and V are both admissible open sets and they give a set-theoretic covering of $\mathrm{Sp}\left(\mathbb{C}_p\left\langle T\right\rangle\right)$, but they do not form an admissible covering. This can be seen after passing to adifications, because both of U^{ad} and V^{ad} do not contain the point $x_{\mathbb{D}}^-$.

In fact, we can now see that $\mathbb{D}$ is connected. Indeed, this follows from $\mathcal{O}_{\mathbb{D}}(\mathbb{D}) = \mathbb{C}_p\left\langle T\right\rangle$ upon observing that $\mathbb{C}_p\left\langle T\right\rangle$ has no idempotents.

3. Condition 3 allows us to view formal schemes of complete noetherian rings as adic spaces. Here a formal scheme $\mathrm{Spf}(A)$ is mapped to $\mathrm{Spa}(A, A)$. Moreover, recall the "generic fiber" construction of Raynaud/Berthelot which assigns a rigid analytic space to formal scheme which is topologically of finite type over $\mathbb{Z}_p$. For example, classically one constructs the rigid open unit disc by associating to $\mathrm{Spf}(\mathbb{Z}_p[[T]])$ the rigid analytic space cut out by $\mathbb{Z}_p[[T]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. This then makes $\mathbb{Z}_p[[T]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ into the ring of bounded functions on the open unit disc, but there are many more such functions, so this is of course not an affinoid.

In the world of adic spaces, this construction becomes much more natural. There is a notion of fiber product for adic spaces, and then this construction is literally taking the generic point. The example of the unit disc can be explained as follows. The formal scheme $\mathrm{Spf}(\mathbb{Z}_p[[T]])$ is replaced by the adic space $\mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]])$, and its fibered over $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, which was described in Example 2.1.2. Then if $\eta = \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ is the generic point of $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, the adification of the rigid open unit disc is equal to $\mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]]) \times_{\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. This space is not affinoid: for example, it's not even quasi compact.

4. Here is an example due to Buzzard and Verberkmoes where $\mathrm{Spa}(A, A^+)$ is not sheafy. Take

$$A = \mathbb{Q}_p[T, T^{-1}, Z]/Z^2,$$

which is Huber (and even Tate) with $A_0 = \mathbb{Z}_p[p^nT^{\pm n}, p^{-n}T^{\pm n}Z]$, given the p-adic topology. Let $U = \{x : |T(x)| \leq 1\}$ and $V = \{x : |T(x)| \geq 1\}$, so that $U \cup V$. To show $\mathcal{O}_X$ is not a sheaf, it suffices to show that the map $\mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ is not injective. We shall show Z is nonzero in $\mathcal{O}_X(X)$ but zero in $\mathcal{O}_X(U)$ and in $\mathcal{O}_X(V)$. We have $\mathcal{O}_X(X) = \widehat{A}$, so for Z to be 0 it must lie in $\cap p^m A_0$, which it does not because it does not lie even in pA_0. On the other hand, $\mathcal{O}_X(U) = A_0\left\langle T\right\rangle$ and $\mathcal{O}_X(V) = A_0\left\langle T^{-1}\right\rangle$. In $A_0\left\langle T\right\rangle$ we have $Z = p^nT^n\left(p^{-n}T^{-n}Z\right) \to 0$, and in $A_0\left\langle T^{-1}\right\rangle$ we have $Z = p^nT^{-n}\left(p^nT^nZ\right) \to 0$.

4. Perfectoid spaces

In this section we assume the reader has some familiarity with perfectoid rings and tilting. The following definition, from the lecture notes of Scholze and Weinstein, is more general than the original one given in Scholze's thesis in that there is no requirement for a fixed base field.

Definition 4.1. 1. A complete Tate ring is *perfectoid* if it is uniform, and if there exists a pseudounifromizer ϖ such that $\varpi^p | p$ and the Frobenius map $R^\circ/\varpi \to R^\circ/\varpi^p$ is an isomorphism.

2. A *perfectoid Huber pair* is a Huber pair (R, R^+) where R is perfectoid.

Recall there is a notion of tilting which is well behaved for perfectoid rings.

Theorem 4.1. *The functor $S \mapsto S^\flat$ gives an equivalence of categories between perfectoid R-algebras and perfectoid $R^\flat$ algebras.*

This functor is also well behaved with respect to integral subrings.

Proposition 4.1. *The set of rings of integral elements $R^+ \subset R^\circ$ is in bijection with the set of rings of integral elements $R^{\flat+} \subset R^{\flat\circ}$, with the bijection given by $R^{\flat+} = \lim_{x \to x^p} R^+$.*

Thus if $X = \mathrm{Spa}(R, R^+)$ is a perfectoid Huber pair, so is $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$. They are then related by the following theorem.

Theorem 4.2. *Let R be a perfectoid ring with tilt $R^\flat$. There is a homemorphism*

$$X \cong X^\flat, x \mapsto x^\flat,$$

Where $|f(x^\flat)| = |f^\sharp(x)|$. This homeomorphism preserves rational subsets. For any rational subset $U \subset X$ with image $U^\flat \subset X^\flat$, the complete Tate ring $\mathcal{O}_X(U)$ is perfectoid with tilt $\mathcal{O}_{X^\flat}(U^\flat)$.

In particular, each $\mathcal{O}_X(U)$ is uniform. Using Theorem 3.1.4, we get the following.

Corollary 4.1. *A perfectoid Huber pair is sheafy.*

This allows one to define perfectoid spaces as the full subcategory of adic spaces which are locally isomorphic to $\mathrm{Spa}(R, R^+)$ for perfectoid R.

Example 4.1. Suppose R is a perfectoid algebra over a perfectoid field K with psuedouniformizer ϖ which admits arbitrary p-power roots of unity. Scholze's arguments in his thesis show that any rational subset $U^\flat = U\left(\frac{f_1,\dots,f_n}{g}\right)$ of $R^\flat$ can be taken with f_n being a power of $\varpi^\flat$. Moreover, he shows that in that case the inverse image of $U^\flat$ under $X \to X^\flat$ is $U = U^{\flat\sharp} = U\left(\frac{f_1^\sharp,\dots,f_n^\sharp}{g^\sharp}\right)$, and that

$$\mathcal{O}_X(U) = R^\circ \left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle [\varpi^{-1}],$$

which is perfectoid. Finally, every rational subset of X is of this form.

Finally, we discuss Scholze's similarity relation $\sim$.

Definition 4.2. Let X be a perfectoid space over a perfectoid field K, and let $\{X_i\}_{i \in I}$ be a filtered inverse system of noetherian adic spaces over K. Let $\varphi_i : X \to X_i$ be a map to the inverse system. Then $X \sim \varprojlim X_i$ if the map of topological spaces $|X| \to \varprojlim |X_i|$ is a homeomorphism and if for each $x \in X$ with images $x_i \in X_i$, the map

$$\varinjlim k(x_i) \to k(x)$$

has dense image.

The point is that with this notion X behaves as the inverse limit of these spaces even though it may not literally be the inverse limit in the category of adic spaces. Moreover, X seems to behave like $\varprojlim X_i$ in terms of topology, whether it be the analytic (in which case it is obvious) or the etale topology (in which case it is a theorem of Scholze). As to why this can be useful, consider the following example which allows us to pass from characteristic 0 to characteristic p.

Example 4.2. Let $\mathbb{A}_K^{1,\text{perf}}$ be the perfectoid line over K. i.e., it is the union of closed perfectoid discs of radii tending to ∞. Then one can show that

$$\mathbb{A}_K^{1,\text{perf}} \sim \lim_{\leftarrow \varphi} \mathbb{A}_K^{1,\text{ad}}.$$

Using the tiliting equivalence, the topological (analytic or etale) properties of $\mathbb{A}_K^{1,\text{perf}}$ are the same as those of $\mathbb{A}_{K^\flat}^{1,\text{perf}}$. As $K^\flat$ is also perfectoid, we have

$$\mathbb{A}_{K^\flat}^{1,\text{perf}} \sim \lim_{\leftarrow \varphi} \mathbb{A}_{K^\flat}^{1,\text{ad}}.$$

But the Frobenius map φ is purely inseparable in characteristic p, so it has no affect on the etale topology. Therefore this process allows us to identify the etale topology of $\mathbb{A}_K^{1,\text{perf}}$ with that of $\mathbb{A}_{K^\flat}^{1,\text{ad}}$.