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Research Statement

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My research interests are analysis, partial differential equations, fluid mechanics and, more specifically, contour dynamics in incompressible flows. In particular I have studied free boundary problems given by two incompressible fluids with different properties modeled by the 2D surface quasi-geostrophic equation, Darcy's law and Euler's equation, addressing fundamental questions like well-posedness, ill-posedness and long-time behavior.

In the next three sections I will show an introduction of these problems, my main contributions and my future research plans. In the first section I will discuss the dynamics of a sharp front for the 2D surface quasi-geostrophic equation. In the second section I will describe fluid interface problems modeled by Darcy's law which are known in the literature as Hele-Shaw and Muskat problems. In the third section I will consider two contour dynamics problems given by Euler's equation: the motion of a vortex sheet that evolves by the Birkhoff-Rott integral and the free boundary problem between two incompressible fluids with different densities.

A sharp front for QG

The 2D quasi-geostrophic equation (QG) has the following form:

$$\begin{aligned} &\theta_t + u\cdot\nabla\theta = 0, \quad x\in\mathbb{R}^2,\\ &u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{1/2}\psi = -\Lambda\psi, \end{aligned} \tag{1}$$

where $\theta=\theta(x,t)$ is a scalar function that represents the temperature, $u(x,t)$ is the velocity field, and ψ is the stream function. The non-local operator $(-\Delta)^{1/2}=\Lambda$ is defined via the Fourier transform by $((-\Delta)^{1/2}g)\hat{}(\xi) = |\xi|\hat{g}(\xi)$, where $\hat{g}$ is the Fourier transform of g.

This equation has applications in meteorology and oceanography, and is a special case of a more general 3D quasi-geostrophic equation [40]. There has been a great deal of scientific interest in the behavior of the QG equation because it is a plausible model to explain the formation of fronts of hot and cold air.

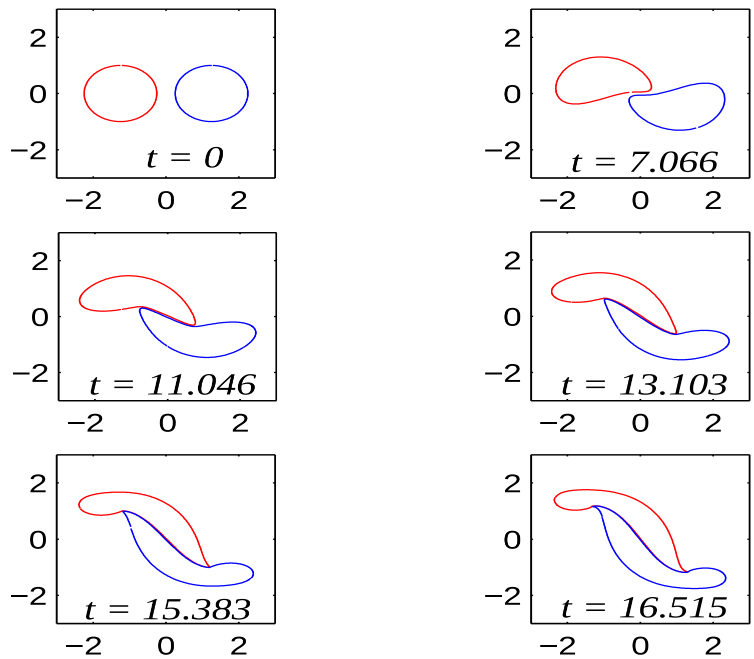
In a different direction Constantin, Majda and Tabak [15] proposed this system as a 2D model for the 3D vorticity intensification Euler equation. They showed that there is a geometric and analytic analogy with 3D Euler. It is not known at this moment if QG can produce singularities.

Resnick [41] studied the weak formulation of the QG equation, and proved existence using Galerkin approximations. The problem of uniqueness is still open.

A fundamental property of QG is that the level sets move with the flow, i.e. that there is no transfer of flow along a level set. Then a natural weak solution, with finite energy, is a connected bounded region $\Omega(t)$ where the function θ satisfies

$$\theta(x,t) = \begin{cases} \theta^1 & \text{if } x\in\Omega(t)\\ \theta^2 & \text{if } x\in\mathbb{R}^2-\Omega(t), \end{cases} \tag{2}$$

with θ^1 and θ^2 constants. This is a model for the evolution of a front of temperature which evolves with a velocity preserving the initial area, $|\Omega(0)|=|\Omega(t)|$.



The equation for the contour reads

$$z_t(\alpha,t) = \frac{\theta^2-\theta^1}{2\pi}\int_{\mathbb{T}} \frac{\partial_\alpha z(\alpha,t)-\partial_\alpha z(\alpha-\beta,t)}{|z(\alpha,t)-z(\alpha-\beta,t)|}d\beta, \tag{3}$$

where $\partial\Omega(t) = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) \,:\, \alpha\in[-\pi,\pi]=\mathbb{T}\}$. They are called *patches* and have been studied for the 2D incompressible Euler equation (see [13] and [8]), where the vorticity is conserved along trajectories.

The evolution of a sharp front for QG was first considered by Resnick in [41]. The velocity u is related to the temperature θ as $u(x,t) = (-R_2(\theta), R_1(\theta))(x,t)$, being R_j the j-th Riesz transform [49]. These operators are Calderón-Zygmund singular integrals and therefore the velocity field of a *patch* for QG is in BMO [49]. Indeed, for a temperature θ satisfying (2), the velocity blows up logarithmically at the free boundary $z(\alpha,t)$. Thus, there is difficulty even in getting the evolution problem. This issue was solved by Resnick since he found that the normal component of the velocity is well-defined for a regular contour (see [42] or [31]). For this type of free boundary problem the tangential component of the velocity does not modify the shape of the interface (in this case it is infinity), and the evolution problem is given by the normal component [34].

Rodrigo in [42] gave a closed system for the *patch* problem for QG where the front is represented by a function. He proved local-existence for a periodic and infinitely differentiable contour by using the Nash-Moser iteration. This tool was used for infinitely differentiable initial data because the operator involved in (3) loses two derivatives.

Córdoba, Fontelos, Mancho and Rodrigo [21] shown evidence of singularities in sharp fronts for QG. More specifically, they give initial data in which the curvature blows up in numerical simulations due to two patches collapsing in a point in a self-similar way (see figure above).

In my Ph.D. thesis (see also [31]), a proof of local-existence of a front convected by the QG equation is presented in Sobolev spaces. With these results we find a justification of the numerical simulations in [21]. The free boundary is not parametrized as a function, therefore the fact that the interface collapses leads to a singularity in the fluid. The local-existence result is obtained by getting local control of the evolution of the following quantity:

$$\mathcal{F}(z)(\alpha,\beta,t)=\frac{|\beta|}{|z(\alpha,t)-z(\alpha-\beta,t)|}\quad\forall\,\alpha,\beta\in[-\pi,\pi],\quad \mathcal{F}(z)(\alpha,0,t)=\frac{1}{|\partial_\alpha z(\alpha,t)|},\tag{4}$$

which measures the arc-chord condition of the curve. This is crucial due to the fact that the operators involved in the equations are ill-defined otherwise. Also, we have to modify the contour equation (3) as follows:

$$z_t(\alpha,t)=\frac{\theta^2-\theta^1}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha z(\alpha,t)-\partial_\alpha z(\alpha-\beta,t)}{|z(\alpha,t)-z(\alpha-\beta,t)|}d\beta+c(\alpha,t)\partial_\alpha z(\alpha,t).\tag{5}$$

A curve that satisfies this new equation also yields a solution of the *patch* problem for QG: we have introduced a tangential term $c(\alpha,t)$ which changes the parametrization of the interface but not the shape. Choosing $c(\alpha,t)$ in an wise fashion, enables the length of the tangent vector to the curve $z(\alpha,t)$ be a function in the variable t only: $A(t)=|\partial_\alpha z(\alpha,t)|^2$ (see [34]). With this property we obtain the following two identities:

$$\partial_\alpha^2 z(\alpha,t)\cdot\partial_\alpha z(\alpha,t)=0,\qquad \partial_\alpha^3 z(\alpha,t)\cdot\partial_\alpha z(\alpha,t)=-|\partial_\alpha^2 z(\alpha,t)|^2.\tag{6}$$

The first equality gives extra cancellation in the system (5) and the second one is a kind of ad hoc integration by parts. We obtain the following theorem:

Theorem 1. *Let $z_0(\alpha)\in H^k(\mathbb{T})$ for $k\geq 3$ with $\mathcal{F}(z_0)(\alpha,\beta)<\infty$ (arc-chord condition). Then there exists a time $T>0$ so that there is a weak solution to QG given by* (2) *where $\partial\Omega(t)=\{z(\alpha,t),\alpha\in\mathbb{T}\}$, with $z(\alpha,t)\in C^1([0,T];H^k(\mathbb{T}))$, and $z(\alpha,0)=z_0(\alpha)$.*

In future research we plan to study possible blow-up criterions for the sharp front in QG and so clarify which are the quantities that must become infinite to obtain singular solutions. It is our intention to get for this kind of contour dynamics equation an equivalent result to the Beale-Kato-Majda blow up criterion for Euler's equation [6]. In this free boundary problem a singularity in the interface appears when the contour collapses and the arc-chord condition is not satisfied. This behavior cannot be found, for example, in the patch problem with the 2D Euler equation [13] [8]. We hope to obtain a blow up criterion in terms of the curvature $\kappa(\alpha,t)$ of the free boundary and the arc-chord condition to authenticate the numerical experiment in [21].

Hele-Shaw and Muskat problems

The evolution of fluids in porous media is an important topic in fluid mechanics encountered in engineering, physics and mathematics [7]. This phenomena has been described using the experimental Darcy's law that, in two dimensions, is given by the following momentum equation:

$$\frac{\mu}{\kappa}v=-\nabla p-(0,\mathrm{g}\rho),$$

where v is the incompressible velocity, p is the pressure, μ is the dynamic viscosity, κ is the permeability of the isotropic medium, ρ is the liquid density, and g is the acceleration due to gravity.

The Muskat problem [39] models the evolution of an interface between two fluids with different viscosities and densities in porous media by using Darcy's law. This problem has been considered extensively without surface tension, in which case the pressures of the fluids are equal on the interface. Saffman and Taylor [44] made the observation that the one phase version (one of the fluids has zero viscosity) was also known as the Hele-Shaw cell equation [32], which, in turn, is the zero-specific heat case of the classical one-phase Stefan problem.

The problem considers fluids with different constant viscosities μ^1, μ^2, and densities ρ^2, ρ^1. Therefore using Darcy's law, we find that the vorticity is concentrated on the free boundary $z(\alpha,t)$, and is given by a Dirac distribution as follows:

$$w(x,t)=\varpi(\alpha,t)\delta(x-z(\alpha,t)),$$

with $\varpi(\alpha,t)$ the vorticity strength. Then $z(\alpha,t)$ evolves with a velocity field coming from the Biot-Savart law, which can be explicitly computed and is given by the Birkhoff-Rott integral of the amplitude ϖ along the interface curve:

$$BR(z,\varpi)(\alpha,t)=\frac{1}{2\pi}PV\int\frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^{2}}\varpi(\beta,t)d\beta \tag{7}$$

where $x^{\perp}=(-x_2,x_1)$ and PV denotes principal value. Using Darcy's law, we close the system with the following formula:

$$\varpi(\alpha,t)=(I+A_{\mu}T)^{-1}\big(-2\mathrm{g}\kappa\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_{\alpha}z_2\big)(\alpha,t), \tag{8}$$

where

$$T(\varpi)=2BR(z,\varpi)\cdot\partial_{\alpha}z,\quad A_{\mu}=(\mu^2-\mu^1)/(\mu^2+\mu^1). \tag{9}$$

Baker, Meiron and Orszag [4] shown that the adjoint operator T^{*}, acting on ϖ, is described in terms of the Cauchy integral of ϖ along the curve $z(\alpha,t)$, and whose real eigenvalues have absolute value strictly less than one. This yields that the operator $(I+A_{\mu}T)$ is invertible and therefore the equation (8) gives an appropriate contour dynamics problem.

The first important question to be asked is whether local-existence is guaranteed. However such a result turns out to be false for general initial data. Rayleigh [43], Taylor [51] and Saffman-Taylor [44] gave a condition that must be satisfied in order to have a solution locally in time, namely that the normal component of the pressure gradient jump at the interface has to have a distinguished sign. This is known as the Rayleigh-Taylor condition. Siegel, Caflish and Howison [48] proved ill-posedness in a 2-D case when this condition is not satisfied (unstable case). On the other hand, they showed global-in-time solutions when the initial data is near planar and the Rayleigh-Taylor condition holds initially. In the same year Ambrose [1] studied this stable case and got energy estimates for the free boundary assuming that the arc-chord and Rayleigh-Taylor condition are satisfied locally in time (see also [3]). In section before we shown that the arc-chord property has to be satisfied locally in time to get a local-existence result for this kind of contour dynamics equations, since a regular interface could touch itself with order infinity and without satisfying the arc-chord property. Also in those works bounds on the operator $(I+A_{\mu}T)^{-1}$ are not taken into consideration. In the case where the viscosities are the same, the character of the interface as the graph of a function is preserved and in my Ph.D. thesis (see [22] and [23]) this fact has been used to give the first local-existence result and a maximum principle, in the stable case, together with ill-posedness in the unstable situation. Recently in [18] we have obtained local-existence in the 2D case, in the more difficult case when the two fluids have different densities and viscosities. In our proof it is crucial to get control of the norm of the inverse operators $(I+A_{\mu}T)^{-1}$. The arguments rely upon the boundedness properties of the Hilbert transforms associated to $C^{1,\delta}$ curves, for which we need precise estimates obtained with arguments involving conformal mappings, the Hopf maximum principle and Harnack inequalities (see [9] and [26]). We then provide bounds for ϖ obtaining

$$\begin{aligned}\frac{d}{dt}(\|z\|^2_{H^k}+\|\mathcal{F}(z)\|^2_{L^\infty})(t)\leq -K\int_{\mathbb{T}}\frac{\sigma(\alpha,t)}{|\partial_{\alpha}z(\alpha)|^2}\partial^k_{\alpha}z(\alpha,t)\cdot\Lambda(\partial^k_{\alpha}z)(\alpha,t)d\alpha\\ +\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^k})(t),\end{aligned} \tag{10}$$

where $K=\kappa/(2\pi(\mu_1+\mu_2))$, $\mathcal{F}(z)$ is given in (4), $\sigma(\alpha,t)$ is the difference of the gradients of the pressure in the normal direction, and the operator Λ is the square root of the Laplacian. When $\sigma(\alpha,t)$ is positive, there is a kind of heat equation in the above inequality but with the operator Λ in place of the Laplacian. Then, the most singular terms in the evolution equation depend on the Rayleigh-Taylor condition. In order to integrate the system we study the evolution of

$$m(t)=\min_{\alpha\in\mathbb{T}}\sigma(\alpha,t) \tag{11}$$

(see [23]), which satisfies the following bound

$$|m'(t)|\leq\exp C(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^k})(t),$$

for almost every t. Using the pointwise estimate $f\Lambda(f) \geq \frac{1}{2}\Lambda(f^2)$ (see [17]) in estimate (10), we obtain

$$\frac{d}{dt}E_{RT}(t) \leq C \exp C E_{RT}(t),$$

where E_{RT} is the energy of the system given by

$$E_{RT}(t) = \|z\|^2_{H^k}(t) + \|\mathcal{F}(z)\|^2_{L^\infty}(t) + 1/m(t).$$

Here we point out that it is completely necessary to consider the evolution of the Rayleigh-Taylor condition to obtain bona fide energy estimates (see [18] and [19]). Finally we have the following result:

Theorem 2. *Let $z_0(\alpha) \in H^k(\mathbb{T})$ for $k \geq 3$, $\mathcal{F}(z_0)(\alpha,\beta) < \infty$, and $\sigma(\alpha, 0) > 0$ (Rayleigh-Taylor condition). Then there exists a time $T > 0$ so that there is a solution to the Muskat problem for fluids with different densities and viscosities with $z(\alpha, t) \in C^1([0, T]; H^k(\mathbb{T}))$ and $z(\alpha, 0) = z_0(\alpha)$.*

Currently we are studying the long-time behavior of a particular case of the Muskat problem. For fluids of the same viscosities, by means of Darcy's law, we can get the following formula for the difference of the gradients of the pressure in the normal direction: $\sigma(\alpha, t) = \mathrm{g}(\rho^2 - \rho^1)\partial_\alpha z_1(\alpha, t)$. A wise choice of parameterizing the curve is that for which we have $\partial_\alpha z_1(\alpha, t) = 1$ [22], getting the Rayleigh-Taylor condition for all time. An additional advantage is that we avoid a kind of singularity in the fluid when the interface collapses due to the fact that we can take $z(\alpha, t) = (\alpha, f(\alpha, t))$ and therefore $\mathcal{F}(z)(\alpha,\beta) \leq 1$ obtaining the arc-chord condition for all time. We have proved [23] that the L^∞ norm of any solution for this case decays, and numerical simulations of the dynamics of the interface show a regularity effect [24] (see Figure 1 below).

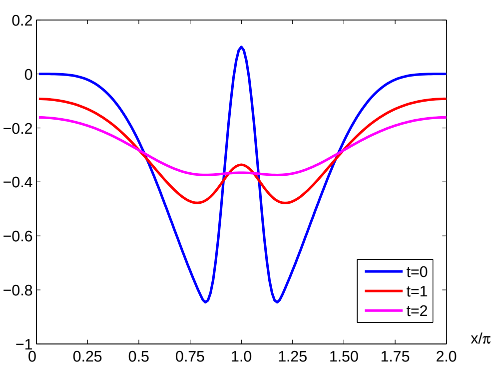


Figure 1: Evolution of the interface $z(\alpha, t) = (\alpha, f(\alpha, t))$.

Also we plan to extend the result of local-existence for the case of fluid with different densities and viscosities in 3D. Now the fluid flow is modeled by Darcy's law in three dimensions. This is an open problem because in work [2] there is no consideration, among other things, of bounds of the analogous of the operators $(I + A_\mu T)^{-1}$ in 3D.

Evolution of an interface by Euler equations

In this section we describe two different contour dynamics problems without surface tension given by Euler's equation. We consider the motion of a vortex sheet that evolves by the Birkhoff-Rott equations and the free boundary problem between two incompressible fluids with different densities.

The vortex sheet equation provides an outstanding kind of weak solution of the two dimensional Euler's equation

$$v_t + (v \cdot \nabla)v = -\nabla p, \quad \nabla \cdot v = 0,$$

where the evolution of an interface between two immiscible fluids of the same density is modeled in such a way that the vorticity $\omega = \nabla \times v$ is concentrated on the free boundary as a Dirac mass as follows:

$$\omega(x,t) = \varpi(\alpha,t)\delta(x - z(\alpha,t)). \tag{12}$$

The function $\varpi(\alpha,t)$ is the vorticity strength and the vortex sheet $z(\alpha,t)$ evolves satisfying the equation

$$z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t), \tag{13}$$

where the Birkhoff-Rott integral is given by (7) and $c(\alpha,t)$ again represents the re-parametrization freedom. Then we can close the system using Bernoulli's law with the equation:

$$\varpi_t = \partial_\alpha(c\,\varpi). \tag{14}$$

We consider initial vorticity strength with zero mean which is preserved by equation (14). This is necessary to obtain a velocity field in L^2 [12].

The problem of existence of weak solutions of the Euler equations for a general initial velocity in L^2 is not well understood [38]. Constantin, E and Titi [16] prove a condition of regularity in 3D within the chain of Besov spaces, $v \in L^3([0,T], B_3^{\alpha,\infty}) \cap C([0,T], L^2)$ with $\alpha > 1/3$, for weak solutions conserving energy (Onsager's conjecture). Nevertheless there are non-uniqueness results for weaker solutions with zero initial data that become nontrivial (see [45] and [47]) even for velocity fields in L^2, i.e. $v(x,t) \in L_c^\infty([0,T] \times L^2)$ (see [36]). There is also a uniqueness result for a vorticity in $L^1 \cap L^\infty$ due to Yudovich [55].

For the particular case of a vortex sheet there are many papers which consider the case of ϖ with a distinguished sign. We can point out the work of Delort [27] where he proved global existence of weak solutions for initial velocity in L^2_{loc} and vorticity a positive Radon measure. In the case of analytic initial data, a local existence result for the vortex sheet is given by Sulem, Sulem, Bardos and Frisch in [50]. Duchon and Robert [28] prove global-existence for a peculiar initial interface. Caflish and Orellana [10] also show global-existence for similar particular initial data and moreover they give an argument to prove ill-posedness. A study of the existence of solutions of this problem in less regular spaces than H^s can be found in [52].

In the particular case of a vortex sheet with distinguished sign the solutions of the system are waves whose Fourier modes grow like $e^{\pi|\xi|t}$ and $e^{-\pi|\xi|t}$. This phenomena is called the Kelvin-Helmholtz instability (see [35] and [33]). The peculiar solution given by Duchon and Robert are those that only stimulate the dissipative waves with growth $e^{-\pi|\xi|t}$. Caflish and Orellana used some properties of the equation to give solutions that stimulate the modes which grow like $e^{\pi|\xi|t}$. Then they obtain ill-posedness in Sobolev spaces in the Hadamard sense.

In the recent work [12] we were able to prove ill-posedness for the case of zero mean vorticity strength. We consider the equations for the evolution of the vortex sheet with $c(\alpha,t) = H(\varpi)(\alpha,t)$ where H is the Hilbet transform operator (see [49]). For a planar curve $z(\alpha,t) = (\alpha,0)$, we have exactly

$$BR(z,\varpi)(\alpha,t) = \frac{1}{2}H(\varpi)(\alpha,t)(0,1)$$

and therefore the parameter $c(\alpha,t)$ is just of the same order as the Birkhoff-Rott integral over ϖ for a regular interface. Then we show ill-posedness for the equation of the amplitude (14) for non-analytic initial data with zero mean (see also [11]).

Theorem 3. *Let $\varpi_0 \in H^s$ with $s > \frac{3}{2}$ and zero mean. Then if there exists a point α_0 where $\varpi_0(\alpha_0) > 0$ and ϖ_0 is not C^∞ in α_0, there is no solution of equation (14) with $c = H(\varpi)$ in the class $C([0,T); H^s)$, $s > \frac{3}{2}$ and $T > 0$. In addition, $\varpi_0 \in C^\infty$ is not sufficient to obtain existence.*

The water waves problem treats the evolution of a free boundary given by air and water, governed by the Euler equation, and considered with the force of gravity. This problem considers the air and water with densities equal to zero and one respectively. It reads

$$\begin{aligned} &\rho_t + (v\cdot\nabla)\rho = 0, \qquad \nabla\cdot v = 0, \\ &\rho(v_t + (v\cdot\nabla)v) = -\nabla p - (0, \mathrm{g}\rho), \end{aligned} \tag{15}$$

where v is the fluid velocity, ρ is the density, p is the pressure, g is the acceleration due to gravity and the vorticity is given by a delta function as in (12). The equation of the interface evolution is given by (13) and we close the system using Bernoulli's law with the formula

$$\begin{aligned}\varpi_t = &-2A_\rho \partial_t BR(z,\varpi)\cdot\partial_\alpha z - A_\rho \partial_\alpha\Big(\frac{|\varpi|^2}{4|\partial_\alpha z|^2}\Big) + \partial_\alpha(c\,\varpi)\\ &+ 2A_\rho c\partial_\alpha BR(z,\varpi)\cdot\partial_\alpha z + 2A_\rho \mathrm{g}\partial_\alpha z_2,\end{aligned} \tag{16}$$

where $A_\rho = (\rho^2-\rho^1)/(\rho^2+\rho^1)$ is the Atwood number and ρ^1, ρ^2 are any constant densities.

Also for this free boundary problem the Rayleigh-Taylor condition has to be satisfied in order to have a solution locally in time. When that condition is not imposed initially the evolution equation is ill-posed (see [29] and [30]). Beale, Hou and Lowengrub [5] showed that the linearization of the water wave equation is well-posed provided the initial data satisfy the Rayleigh-Taylor condition. The well-posedness was first obtained by Wu (see [52] and [53]). She proved that the presence of the gravitational field, together with the hypothesis about the asymptotic flatness of the fluid domains, implies that the Rayleigh-Taylor signum condition must hold so long as the interface is well-defined (see [52] The Key Lemma pg. 45). We also cite [3] where the authors get energy estimates on the free boundary and the amplitude of the vorticity, under the time dependent assumption of the arc-chord property. They also make use of the fact obtained by Wu regarding the persistence of the Rayleigh-Taylor sign condition.

Recently [19] we have obtained local existence for multiple cases: with or without gravity, but also for a closed boundary or a periodic boundary with the fluids placed above and below it. In our work, we consider an initial interface satisfying Rayleigh-Taylor and the arc-chord conditions. It is part of the evolution problem to check that such properties prevail for a time depending conveniently upon the initial data. Let us note that equation (16) can be written in the following more convenient explicit manner:

$$\varpi_t(\alpha,t) = (I + A_\rho T)^{-1}(A_\rho R(z,\varpi) + \partial_\alpha(c\varpi))(\alpha,t), \tag{17}$$

for T given by (9). Therefore it becomes crucial to get explicit bounds on the operator $(I + A_\rho T)^{-1}$ as mentioned earlier. We consider the following definition of energy

$$\begin{aligned}E_{RT}(t) = &\|z\|^2_{H^{k-1}}(t) + \int_{-\pi}^{\pi}\frac{\sigma(\alpha,t)}{\rho^2|\partial_\alpha z(\alpha,t)|^2}|\partial_\alpha^k z(\alpha,t)|^2 d\alpha + \|\mathcal{F}(z)\|^2_{L^\infty}(t)\\ &+ \|\varpi\|^2_{H^{k-2}}(t) + \|\varphi\|^2_{H^{k-\frac12}}(t) + 1/m(t),\end{aligned} \tag{18}$$

where $\varphi = \varpi/(2|\partial_\alpha z|) - c|\partial_\alpha z|$ (see [5] and [3]). Then we obtain

$$\frac{d}{dt}E_{RT}(t) \leq \exp(CE_{RT}(t)), \tag{19}$$

and it yields the following theorem.

Theorem 4. *Let $z_0(\alpha) \in H^k$, $\varphi_0(\alpha) \in H^{k-\frac12}$ and $\varpi_0(\alpha) \in H^{k-1}$ for $k \geq 4$, $\mathcal{F}(z_0)(\alpha,\beta) < \infty$, and $\sigma(\alpha,0) > 0$ (Rayleigh-Taylor). Then there exists a time $T > 0$ so that we have a solution to the water wave problem ($\rho^1 = 0$), where $z(\alpha,t) \in C^1([0,T];H^k)$ and $\varpi(\alpha,t) \in C^1([0,T];H^{k-1})$ with $z(\alpha,0) = z_0(\alpha)$ and $\varpi(\alpha,0) = \varpi_0(\alpha)$.*

Finally we cite the following interesting works by Christodoulou-Lindblad [14], Lindblad [37], Coutand-Shkoller [25], Shatah-Zeng [46] and Zhang-Zhang [56] where the rotational case has also been considered. We point out that the evolution of the sign of the Rayleigh-Taylor condition is crucial in estimate (19), because it allows us to get rid of the highest order derivatives in the evolution equation of the Sobolev norms of the curve.

In the interface problem between fluids of different densities, the singular term $\partial_\alpha(c\,\varpi)$ is in the evolution equation (16). We recall the ill-posed problem for the vortex sheet evolution equation, where the Kelvin-Helmontz instability appears in equation (14). It is shown in [5] [3] that in the particular case of $A_\rho = 1$ ($\rho^1 = 0$ and therefore it is a fluid in vacuum) this bad term can take part of a simpler transport quantity when the evolution equation for the parameter $\varphi = \varpi/(2|\partial_\alpha z|) - |\partial_\alpha z|c$ is taken into consideration (see also [20]). Then for $0 < A_\rho < 1$ local-existence or ill-posedness is an open problem when we consider initial data satisfying the Rayleigh-Taylor condition. We plan to study these important cases.

In our recent work [20] we have understood the water wave problem as a weak solution of the Euler equation (15) where the vorticity is given by a delta function. We aim at extending the local-existence result to the rotational case and the irrotational and rotational cases in 3-D.

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