Higher Derivatives

If \( f' \) exists on an interval around a point \( x \), we can define \( f''(x) \) etc.

with derivative \( f''(x) \).

\( f \) is \( n \) times differentiable at \( x \) if it has \( (n-1) \) derivatives existing in a neighborhood of \( x \) and \( f^{(n-1)} \) is differentiable at \( x \).

Taylors Theorem Suppose \( f: [a,b] \rightarrow \mathbb{R} \), new, \( f^{(n-1)} \) is continuous on \( [a,b] \) and \( f^{(n)}(t) \) exists for all \( t \in [a,b] \). Then for every \( x, x_0 \in [a,b] \)

\[
f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(t)}{n!}(x-x_0)^n,
\]

for some \( t \) between \( x, x_0 \).
Proof:

eq f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k \\
\text{(n-1) order Taylor polynomial}

\text{Call } M \text{ s.t. } f(t) = f(t) - P(t) - M(t-x_0)^n

\text{and } g(t) = f(t) - P(t) - M(t-x_0)^n

\text{Note that } f(t) = \int g(t) dt

\text{and } g(x_0) = f(x_0) - P(x_0) - M(x-x_0)^n

\text{and } g^{(n)}(x_0) = f^{(n)}(x_0) - f^{(n)}(x_0) = 0

\text{so } g(x) = \int g(t) dt

\text{and } M \text{ chosen s.t. } g(x) = 0

\Rightarrow \exists x_1 \in (x_0, x) \text{ s.t. } g'(x_1) = 0

\text{then } g'(x_0) = g'(x_1) = 0 \Rightarrow \exists x_2 \in (x_0, x_1)

\Rightarrow \text{ w/ } g''(x_2) = 0
\[ \exists x_k e (x_0, x_k-1) \ \text{ s.t. } g^{(k)}(x_k) = 0 \]

for \( k = 1, 2, \ldots, n \)

but \( g^{(n)}(x_n) = 0 \)

\[ \Rightarrow f^{(n)}(x_n) = n5M \]

\[ \Rightarrow M = \frac{f^{(n)}(x_n)}{n} \]

**Differentiation of Vector-Valued Functions on \( \mathbb{R} \)**

We can apply the same definition of differentiability for vector-valued functions.

**For example**

\[ f : (a, b) \to \mathbb{C} \]

or \( f : [a, b] \to \mathbb{R}^k \)

\( f'(x) \) **is defined**: \( \forall x \in (a, b) \)

\[ \lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = 0 \]
Note that this holds iff each component converges:

\[ \lim_{t \to x} \frac{f_j(t) - f_j(x)}{t - x} \to \frac{f'_j(t) - f'_j(x)}{t - x} \quad \text{as} \ t \to x \]

e.g. \( f : \mathbb{R^k} \to \mathbb{R} \) is diff. at \( x \in \mathbb{R^k} \)

iff \( 1 \leq j \leq k \) the component \( f_j \) is diff. at \( x \) and

\[ \frac{f'(x)}{= (f'_1(x), \ldots, f'_k(x))} \]

Then \( f \) diff. \( \iff \) \( g \) diff.

Then \( (f + g)' = f' + g' \)

\( (f \circ g)' = f' \circ g + f \circ g' \)

The mean value theorem however does not hold for vector valued \( f \) and \( g \).
Example: \[ f(t) = e^{ix} = \cos x + i\sin x \]

on \( [0, 2\pi] \)

This is a parametrization of the unit circle

\[ |f(t)|^2 = \cos^2 x + \sin^2 x = 1 \quad \text{for all } x \in \mathbb{R}. \]

\[ f(0) = f(2\pi) = 0 \]

but \( f'(x) = \sin x - i\cos x = ie^{ix} \)

also has \( |f'(x)| = 1 \quad \text{for all } x \)

so \( f'(x) \to 0 \) \( \text{as } x \to 0 \) \( \text{or } 2\pi \)

On the other hand \( |f(x)|^2 \) does satisfy the mean value inequality since it is a map \( \mathbb{R} \to \mathbb{R}_+ \)

which can sometimes be useful.

Even more:

Theorem: Suppose \( f \) is a cts map \( [a,b] \to \mathbb{R}^2 \) and \( f \) is diff on \( (a,b) \). Then \( \exists c \in (a,b) \) s.t.

\[ |f(b) - f(a)| \leq |b-a| |f'(c)|. \]
Proof: Let \( z = f(b) - f(a) \) and define

\[ \phi(t) = z \cdot f'(t) \quad \text{for} \quad a \leq t \leq b \]

This is continuous on \([a,b]\) and differentiable on \((a,b)\) as a product of differentiable functions.

Then \( \phi(b) - \phi(a) = (b-a) \cdot f'(c) = (b-a) \cdot z \cdot f'(c) \) for some \( c \in (a, b) \)

\[ \phi(b) - \phi(a) = z \cdot [f(b) - f(a)] = |f(b) - f(a)|^2 \]

So \( |f(b) - f(a)|^2 = (b-a) \cdot z \cdot f'(c) \)

\[ \leq (b-a) \cdot 2 \cdot |f'(c)| \]

So \( |f(b) - f(a)| \leq (b-a) \cdot |f'(c)| \) for some \( c \in (a,b) \)
Ordinary Differential Equations

ODE initial value problem look for \( y : [t_0, T] \rightarrow \mathbb{R}^k \)

Solving \[ \begin{cases} \dot{y} = f(t, y) \quad \text{for} \quad t \to t_0 \\ y(t_0) = y_0 \end{cases} \]

Questions: 1. Existence of solution
   2. Uniqueness of solution

Theorem: Suppose that \( f : [t_0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}^k \) is Lipschitz as in \( y \) variable.

Then there exists at most one \( y : [t_0, T] \rightarrow \mathbb{R}^k \)

which is \( C^1 \) on \([t_0, T]\) and \( C^k \) on \(([t_0, T])\).

Solving \[ \begin{cases} \dot{y}(t) = f(t, y(t)) \quad \text{for} \quad t \to t_0 \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \]

Proof: Suppose \( y_1 \) and \( y_2 \) both solve

then \[ \begin{cases} (y_1 - y_2)' = f(t, y_1) - f(t, y_2) \\ (y_1 - y_2)(t_0) = 0 \end{cases} \]

by the "mean value property" for vector valued functions

then \( y_1 = y_2 \)
Call $z = y_1 - y_2$

$|z'| = |\phi(t, y_1) - \phi(t, y_2)| \leq A|z|$

Using $\phi$'s Lipschitz in $y$ variable.

Thus $z$ solves

\[
\begin{cases}
|z'| \leq A|z| & \text{for } t \in [0, T] \\
 z(0) = 0
\end{cases}
\]

\underline{LM: Suppose $z$ solves the above eqn. Then $z'(t) = 0$ for all $t \in [0, T]$.}

\underline{Proof:} Call $M = \max_{t \in [0, T]} |z(t)|$

and let $\delta > 0$ small enough so that

$\delta A M < 1$

we show that $z(t) = 0$ for $t \in (0, \text{total})$

then just apply same argument \( \frac{T-\text{total}}{\delta} \) times

by vector MVP for every $t \in [0, \text{total}]$

$\exists t^{*} \in (0, T)$ such that $|z(t^{*})| \leq (T-\text{total})|z(0)|$

$\leq \delta A M |z(0)|$

$\leq \delta A M$
\[ 33 \]

\[ f' = \phi_0 \]

\[ y' = \phi_1 = 0 \]

\[ \int f' = f_{14} \]

We will return to the question of whether this week's event will be a confirmation of the hypothesis. If so, then

\[ M \leq \delta \cdot \overrightarrow{r} \leq \delta' \]

And, in this case, we can assume that

\[ |x - z| = 0 \]

\[ t \neq f(x, y) \]