Sequential compactness for continuous functions

Given a sequence \( f : E \rightarrow \mathbb{R} \) what kind of conditions can we put to get subsequential convergence?

For we certainly need \( f(x) \) to be bounded for each \( x \in E \).

(Sequences of real numbers have convergent subsequences (more or less) in real boundedness)

Def we say \( f \) uniformly bounded on \( E \) if \( \exists M > 0 \) s.t.

\[
|f(x)| \leq M \quad \forall x \in E, \forall n \in \mathbb{N}.
\]
A slightly

If \( f_n \) uniformly hold and \( E \subset \mathbb{R} \) is countable then we can find the converging phases on \( E \).

Subsequence Diagonal argument:

\[ E_1 = \{ x_1, x_2, \ldots \} \]

Let \( S \) be a subset of \( E \).

Let \( \{ f_n(x) \} \) converge.

Let \( \{ f_n(x_1) \} \) be a subseq of \( \{ f_n(x) \} \) and \( \{ f_n(x_2) \} \) converge.

and so on.
Take the diagonal subseq $f_{k,k}
\text{then } f_{k,k}(x_j) \text{ converge as } k \to \infty \text{ for each } j \in \mathbb{N}.

In general this is not enough to make $f$ converge everywhere.

Example $f(x) = \sin(nx)$
Equicontinuity

To get an everywhere convergent subseq we will need to put a stronger assumption.
(no to make an assumption to get)
just put more convergence around the point

Def: A family \( \mathcal{F} \) of functions \( f: X \to \mathbb{R} \) (metric space \( X \)) is called equicontinuous if

\[ \forall \varepsilon > 0 \exists s \in S \text{ s.t. } \forall x, y, z \in X \quad |f_x - f_y| < \varepsilon \text{ for all } \delta > 0 \text{ and all } f \in \mathcal{F}. \]
Theorem If \( K \) is a compact metric space and \( f_n \in C(K) \) s.t. \( f_n \) converges uniformly on \( K \), then \( f_n \) is equicontinuous in \( K \).

Therefore, if \( Q \subseteq C(K) \) is sequentially compact \( \Rightarrow \) equicontinuous and uniformly bounded.

Theorem If \( Q \subseteq C(K) \) is compact then \( Q \) is uniformly bounded and equicontinuous.

Proof: \( B \subseteq C(K) \Rightarrow \) bounded (in sup norm intrinsic case) \( \Rightarrow \) uniformly bounded

\[ \|f\|_{\sup} \leq M \quad \forall f \in Q \]

for some \( M > 0 \)

Let \( \varepsilon > 0 \) and \( f_1, \ldots, f_n \in Q \) s.t. \( f_n \) s.t.

\[ \|f_n - f\|_{\sup} < \varepsilon/3 \quad \text{for all some} \quad i,j \in \mathbb{N} \]
Now \( f_j \) are equicontinuous, \( \forall \varepsilon > 0 \)

so \( \exists \varepsilon \in \mathbb{R}^+ \) such that

\[
\forall x, y \in 
\frac{|f_j(x) - f_j(y)|}{\varepsilon} < \varepsilon \forall \varepsilon \in \mathbb{R}.
\]

Now let \( f \) be arbitrary, \( \forall \varepsilon > 0 \)

and \( f_j \) s.t.

\[
|f_j(x) - f_j(y)| < \varepsilon \forall \varepsilon \in \mathbb{R}.
\]

\[
|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)|
\]

\[
\leq 3 \cdot \frac{\varepsilon}{3} = \varepsilon
\]

\[\Rightarrow f \text{ is equicontinuous family.}\]

In particular, convergent sequences from \( \mathbb{C}^k \) sub to \( f \) if \( f \) is \( \varepsilon \) mit \( \forall \varepsilon \in \mathbb{R} \) and equicontinuous.
If \( f \in C(K) \) is uniformly bounded and equicontinuous then \( f \) is compact in \( C(K) \).

I.e., if sequence \( f_n \) is uniformly bounded and equicontinuous then it has a subsequence \( f_{n_k} \) converging uniformly on \( K \).

This is called the Arzelà–Ascoli Theorem.

One of the most important results of this class.

Proof: Since \( K \) is compact, it is separable.

Let \( E \subset K \) be a countable dense set.

By previous result any sequence \( f \in C(K) \) has a subsequence \( f_{n_k} \) which converges on \( E \).

Let just call this \( f_k \) and forget about original sequence.
Let \( \varepsilon > 0 \) and pick \( \delta \) from the appropriate \( \delta \)-\( \varepsilon \) continuity.

\[ f(x) - f(y) < \varepsilon \quad \text{if} \quad d(x,y) < \delta \]

for all \( x, y \in \mathbb{R}^n \).

Since \( E \) dense, \( K \subseteq \bigcup_{x \in E} B(x, \delta) \)

\( K \) is compact \( \Rightarrow \) \( K \subseteq B(x_1, \delta) \cup \cdots \cup B(x_m, \delta) \)

for some \( x_1, \ldots, x_m \in E \).

Since \( f \) converges on \( E \), there is \( N \in \mathbb{N} \) s.t. \( k \geq N \Rightarrow \)

\[ |f_k(x_i) - f(x_i)| < \varepsilon \quad \forall \ 1 \leq i \leq m. \]

Then for \( x \in K, \ x = x_i \\text{ in } \text{d}(x_i, x) < \varepsilon \)

\[ |f_k(x) - f(x)| \leq |f_k(x) - f_k(x_i)| + |f_k(x_i) - f(x_i)| + |f(x_i) - f(x)| \]

\[ \leq 3\varepsilon \]

so \( |f_k - f|_{\text{sup}} < \varepsilon \) for \( k \geq N \). \( \square \)