Existence for ODE initial value problems

Look for a solution $X_t$ of the following problem

\begin{align}
(1) \quad \begin{cases}
X_t = f(t, X_t) & \text{for } t > 0, \\
X_0 = x_0
\end{cases}
\end{align}

Here $X_t$ is just a notation for $\frac{d}{dt} X_t$

We will prove two versions of the existence theorem.

**Picard's Theorem**

Here we assume that $f(t, \cdot)$ is Lipschitz continuous uniformly in $t \in \mathbb{R}$.

In $\mathbb{E} \subset \mathbb{R}^n$

\begin{align}
(2) \quad \| f(t, X_t) - f(t, Y_t) \| \leq L \| X_t - Y_t \|
\end{align}

and $f$ is in $t$ variable.

Instead of solving the ODE we will solve it in integral form

\begin{align}
(3) \quad X_t = x_0 + \int_0^t f(s, X_s) \, ds
\end{align}

by F76: If $X_t$ solves (3) then $X_t$ is CS.

$\implies$ (by F76 and the formula (3)) that
\( X_t \) is differentiable and

\[
\dot{X}_t = f(t, X_t)
\]

**Theorem:** Suppose that \( f \) satisfies above assumption. Then there exists a global, unique solution to the integral equation (3).

**Proof:** We will argue using the contraction mapping theorem.

Consider the functional \( \Phi : C([0, T]) \to C([0, T]) \)

\[
\Phi(X, t) = x_0 + \int_0^t f(s, X_s) \, ds
\]

defined for every \( C^1 \) path \( X(t) \) in \( \mathbb{R}^n \).

Since \( f(t, X_t) \) is \( C^1 \) \( \Rightarrow \) \( R.I. \)

\( \Phi(X, t) \) is \( C^1 \) in \( t \) since \( X_t \) is \( C^1 \) on \([0, T]\) \( \Rightarrow \) \( \Phi(X, t) \) \( C^1 \) by some M

\[
|\Phi(X)(t) - \Phi(X)(t_0)| = |\int_{t_0}^t f(s, X_s) \, ds| \leq M |t - t_0|
\]
how show $B$ is a contraction for $T$ small.

\[
\left| \mathcal{F}(X_t) - \mathcal{F}(Y_t) \right| = \left| \int_0^T f(s, X_s) - f(s, Y_s) \, ds \right|
\leq \int_0^T \left| f(s, X_s) - f(s, Y_s) \right| \, ds
\leq \int_0^T L \left| X_s - Y_s \right| \, ds
\leq LT \| X-Y \| \exp(LT)
\]

\[\frac{d}{dt} \mathcal{F}(X_t) = \mathcal{F}(X_t) \sup_{[0,T]} = LT \| X-Y \| \sup_{[0,T]}
\]

for $T < \frac{1}{2L} \quad \Rightarrow \quad B$ is a contraction.

so $B$ has a unique fixed pt $X^*$ which solves

\[X^*_t = x_0 + \int_0^T f(s, X_s^*) \, ds = \mathcal{F}(X_t^*) (T)
\]

how to extend $X_t$ from solution on $[-\frac{1}{2L}, \frac{1}{2L}]$ to whole real line just note that can take

$X_{\frac{1}{2L}}$ as initial data at $t = \frac{1}{2L}$ and

have solution on $[\frac{1}{2L}, \frac{1}{2L}]$ etc.
When \( f(t, x) \) is just locally Lipschitz in \( x \)

\( e.g. \) if \( \frac{Df}{dx} \) is cts (i.e., \( \text{odd on open sets} \))

then one has to be more careful then.

\[ \begin{align*}
  X_0 &= x_0 > 0 \\
  x_t &= x_0 \\
  X_t &= \frac{x_0}{1 - x_0 t}
\end{align*} \]

solution is 

\( x_t \) blows up at \( t = \frac{1}{x_0} \).

i.e., no global in time solution.

**Prop**: Suppose \( f \) is cts and \( \frac{Df}{dx} \) is cts \& \( f \) is cts \( x \) then for every \( x_0 \), \( \exists \) a time \( T(x_0) \) s.t. there is a unique \( c \) solution \( \Phi(3) \) on \( [-T(x_0), T(x_0)] \).

**Proof**: We try to do the same contraction mapping argument, but we need to be more careful.

Since \( \frac{Df}{dx} \) cts on any open set \( \Omega \) containing \( x_0 \) and \( M, L > 0 \) s.t. \( \| Df \| \leq L \) if \( f \in \Omega \).
Define the metric space for \( T = \frac{r}{M} \wedge \frac{1}{2L} \)

\[ A = \left\{ x_t : [0, T] \rightarrow \mathbb{R}^n \mid x_0 = x_0, x_t \text{ continuous} \right\} \]

This is a closed subspace of \( C([0, T]) \) complete.

Then define \( \Phi \) as follows

\[ \Phi(x)(t) = x + \int_0^t f(s, x_s) \, ds \]

\[ \Phi(x)(0) = x_0 \]

\[ \Phi(x) \text{ continuous} \]

\[ |\Phi(x) - x_0| \leq \int_0^T |f(s, x_s)| \, ds \leq M^2 \int_0^T |x_s| \, ds \leq M^2 \max_{s \leq T} |x_s| \]

\[ \leq M^2 \int_0^T |x_s| \, ds \leq M^2 T \cdot \max_{s \leq T} |x_s| \]

\[ \Phi : A \rightarrow A \]

as before \( \Phi \) is a contraction

\[ |\Phi(x) - \Phi(y)| \leq \int_0^T |f(s, x_s) - f(s, y_s)| \, ds \]

\[ \leq \int_0^T \max_{s \leq T} |x_s - y_s| \, ds \leq LT \|x - y\|_{\text{sup}[0,T]} \]

\[ \leq \frac{1}{2} \|x - y\|_{\text{sup}[0,T]} \]
Peano Existence Thm

The other direction to extend this result is to remove the Lipschitz requirement.

The issue is that such ODEs no longer have uniqueness so fixed pt argument will not work.

\[
\begin{align*}
\text{examp: } &\quad \begin{cases} 
X_t = X^{1/2} \\
X_0 = 0
\end{cases} \\
\text{family of sols: } &\quad X_t = \left( \frac{1}{4} (t - b_0)^2 \right) + \text{ for } t \geq 0
\end{align*}
\]

Instead of contraction mapping we will use completeness argument, in with contraction mapping we iterate.

\[
\begin{align*}
X^n_t = X_0 + \int_0^t f(s, X^{n-1}_s) \, ds \\
\text{w/ } X_t = x_0
\end{align*}
\]

But now we use eqn (4) to show that itarates are uniformly bounded and equicontinuous.

\[
\Rightarrow \text{ pre-compact in } C([0, T])
\]
Well I think that sequence of chebyshev works, but let's do it the way Rabin suggests (problem 25 chapter 7)

Theorem Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is continuous, then for each $x_0 \in \mathbb{R}^d$ on an interval $[\tau, T]$ and a solution $x(t)$ of the IVP

$$\begin{align*}
\dot{x}(t) &= f(t, x(t)) \\
 x(\tau) &= x_0
\end{align*}$$

proof: We construct a sequence of approximate solutions by the Euler method

Call $K = [-\alpha, \alpha] \times B(x_0, r)$ s.t. $\partial B(x_0, r) \subset \mathbb{R}^d$

Since $f$ is $C^1$ on $K$, it is bounded on $K$.

There is $M > 0$ s.t. $|f(t, x)| \leq M$ on $K$.

Define $T = \min\left( \alpha, \frac{br}{M} \right)$

Let $e > 0$, $\varepsilon > 0$ s.t. $A \left( \varepsilon, x_0 \right)$, $x_1, x_2 \in K$

If

$$|t_1 - t_2| < \varepsilon, \quad |x_1 - x_2| < \varepsilon$$

then

$$|f(t_1, x(t_1)) - f(t_2, x(t_2))| < 3\varepsilon$$

We need $3\varepsilon$ because $f(t, x(t))$ is not necessarily Lipschitz.
divide \([0, T]\) up into intervals of length \(s\).

\[ t_i - t_{i-1} \leq \min \left( \frac{s}{M}, \frac{s}{\delta} \right) \]

Now we define a piecewise linear approximation of an ODE solution by

\[ \dot{X}_t^\varepsilon = f(t, X_t^\varepsilon, X_{t-1}^\varepsilon) \quad \text{for} \quad t_{i-1} < t < t_i \]

i.e.,

Since the slope $\dot{X}_t^\varepsilon = \left| f(t, X_t^\varepsilon, X_{t-1}^\varepsilon) \right| \leq M$ for all $t$ (we assume $f$ is $C^1$ across $t_i$),

\[ \|X_t^\varepsilon\| \leq M t \leq M T \leq \varepsilon \quad \Rightarrow \quad (t, X_t^\varepsilon) \in K \text{ for } t \in [0, T] \]

Furthermore, $X_t^\varepsilon$ satisfies that

\[ \dot{X}_t^\varepsilon = f(t, X_t^\varepsilon, X_{t-1}^\varepsilon) = f(t, X_t^\varepsilon) + (f(t_{i-1}, X_{t_{i-1}}^\varepsilon) - f(t, X_t^\varepsilon)) \]}
Now since $b_i - x_i \leq s_i - v_i \leq s_i - v_i$ and $\Delta s_i = 0$, let $\Delta t_i = 0$.

Thus $X^t = b_i - x_i = (s_i - v_i) + \Delta t_i = s_i - v_i$.

If $b_i - x_i = s_i - v_i \leq M^i \leq s_i - v_i$ then

\[ x_j^t = \begin{cases} f(1, x_j^t) + \Delta x_j^t, & \text{if } x_j^t \leq v_j \\ \infty, & \text{otherwise} \end{cases} \]

and $\Delta x_j^t = 0$.

Finally, let $N_i$ be such that

\[ x_i^t = \sum_{j=1}^{N_i} f(x_j^t) + \Delta x_i^t = \sum_{j=1}^{N_i} f(x_j^t) \\ \text{and } \Delta x_i^t = 0 \]
and \( |A_{ij}| < 2 \Rightarrow \Delta_{ij} \to 0 \text{ uniformly.} \)

So taking limits on both sides, we are allowed to exchange limits and integrals by uniform convergence.

\[
X_t \geq x_0 + \int_0^t f(s, X_s) \, ds \quad \text{for } t \in [0, T]
\]

Thus \( X_t \) is differentiable \((GFC)\) and even \( C^1 \).

\[
\Rightarrow X_t \text{ is } C^1 \text{ on } [0, T]
\]

Now that \( X_t \) is \( C^1 \) we can differentiate inside,

\[
X_t' = f(t, X_t) \text{ for } t \in [0, T]
\]

\[
X_0 = x_0
\]