The implicit function theorem

Let $f$ be a $C^1$ function $f: \mathbb{R}^2 \to \mathbb{R}$

A level set of $f$ is $\{ f(x, y) = c \}$ for $c \in \mathbb{R}$

Consider for example a level set $f(x, y) = 0$. This can be seek out on

An implicit equation for $y$ in terms of $x$

or $x$ in terms of $y$

If $f(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) \neq 0$

Then locally we can solve for $x(y)$ so

$$f(x(y), y) = 0$$

e.g. Consider $f(x, y) = x^2 + y^2 - 1$

$$f(x, y) = 0$$

$Df(2, 1) = Df(2, 1) > 0$

so cannot solve uniquely for $y$ near $(2, 1)$.

$Df(2, -1) = Df(2, -1) < 0$, so cannot

solve uniquely for $x(y)$ near
The Linear Version

Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}$

Let $A \in \mathcal{L}(\mathbb{R}^{n+m}, \mathbb{R}^n)$ we can split $A$ as

$A_x h = A(h, 0)$ and $A_y k = A(0, k)$

with $A_x \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $A_y \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.

Then $A(h, k) = A_x h + A_y k$.

Proof: If $A \in \mathcal{L}(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and $A_x$ is invertible

Then for each $k \in \mathbb{R}^m$ there is $h \in \mathbb{R}^n$ such that

$A(h, k) = 0$ i.e. $A(h(\cdot), \cdot) = 0$

and $h = -A_x^{-1}A_y k$.

Since $A_x$ is invertible.

$0 = A(h, k) = A_x h + A_y k = 0$ if $h = -A_x^{-1}A_y k$. 

Thus, $A(h, k) = 0$. 

Since $A_x$ is invertible.
**Nonlinear Version**

The (Implicit Function) let \( f \) be \( C^1 \) mapping \( \mathbb{R}^m \times \mathbb{R}^n \) into \( \mathbb{R}^n \) such that \( f(a,b) = 0 \) for some \( (a,b) \in E \).

Call \( A = f'(a,b) \) and assume that \( A \) is invertible.

Then there exists an open subset \( UC \mathbb{R}^m \), \( WC \mathbb{R}^m \) with boundary \( \partial V \), \( b \in W \) for each \( y \in E \) with

\[(x,y) \in V \quad \text{and} \quad f(x,y) = 0\]

If we call this \( x = g(y) \) then \( g : W \rightarrow \mathbb{R}^n \) is \( C^1 \), \( g(b) = a \) and

\[f(g(y),y) = 0 \quad \text{for} \quad y \in W.

And \( g'(b) = -A^{-1}A_y \)

before going into proof let discuss implications
and the relationship with Inverse Function Theorem.
let $f: \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$ at $a$, $f(a) = b$

and $f'(a)$ is non-singular

Then define $h(x, y): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$h(x, y) := f(x) - y$, $h(a, b) = 0$

then $Dh(a, b) = Df(a)$ is invertible so at a point $W$ of $b$ and any $g: W \to \mathbb{R}^n$ be $C^1$ s.t.

$0 = h(g(y), y) = f(g(y)) - y$ i.e. $g$ is inverse of $f$.

and $g'(b) = -(Dh)^{-1} Dh |_{a,b}$

$= -f'(a)^{-1}(-I) = f'(a)^{-1}$

\[\text{Inverse FT } \Rightarrow \text{Implicit FT}\]

let $h: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $h(a, b) = 0$ for some $(a, b) \in \mathbb{R}^n$
Define \( f : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \) by

\[
f(x,y) = (h(x,y), y)
\]

We want to apply inverse \( PT \) to \( f \),

\[
f'(x,y) = \begin{bmatrix}
0 \times h & D_y h \\
0 & I_{m \times m}
\end{bmatrix}
\]

( block matrix )

\[
f'(a,b)
\]

is invertible iff \( \text{D} h \bigg|_{(a,b)} \) is invertible.

If \( \text{D} h \big|_{(a,b)} \) is invertible then

Inverse \( PT \) is \( \mathcal{G} \) a natural open subset

\( U \times W \) of \( (a,b) \) and \( X \times V \) of \( (u,0,b) \)

s.t. \( f : U \times W \to X \times V \) is 1-1 and onto
Then define \( g(y) = \pi_x(f^{-1}(0, y)) \)

where \( \pi_x : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is defined by

\[
\pi_x(x, y) = x
\]

\( y \) is defined as.

\[
f(g(y), y) = (0, y)
\]

i.e. \( h(g(y), y) = 0 \)

since \( g \) is a composition of \( C^1 \) mappings

it is \( C^1 \) and differentiable

\[
h'(g(y), y) = 0
\]

we get

\[
g'(y) = \left[ \begin{array}{c|c} I_{n \times n} & 0 \\ \hline & f'(g(y), y) \end{array} \right]^{-1}
\]

\[
= \left[ \begin{array}{c|c} I_{n \times n} & 0 \\ \hline & Df(g(y), y) \end{array} \right]^{-1}
\]

\[
\begin{pmatrix} 0 & \frac{\partial h}{\partial y_1} \\ \vdots & \vdots \\ 0 & \frac{\partial h}{\partial y_m} \end{pmatrix}
\]

\[
h'(g(y), y)\begin{bmatrix} g'(y) \\ 0 \end{bmatrix} = 0
\]
\[
\dot{y} = h'(g(y), y) \begin{bmatrix} g'(y) \\ I_n \end{bmatrix} = \begin{bmatrix} \text{Dx}(g(y), y), \text{Dy}(g(y), y) \end{bmatrix} \begin{bmatrix} g'(y) \\ I_n \end{bmatrix}
\]

\[
= \text{Dx} g'(y) + \text{Dy} h
\]

so \[
\begin{bmatrix} g'(y) \\ \text{Dy} \end{bmatrix} = \begin{bmatrix} \text{Dx} \end{bmatrix}^{-1} \text{Dy} h
\]

\[
\begin{bmatrix} g'(y) \\ \text{Dy} \end{bmatrix}(y, y) = 0
\]

Applications:

- Contraction mapping principle: Existence and uniqueness of solutions to systems of ODEs.

This will have to wait till the course on integration.