Problem 1. The general existence and uniqueness theory for scalar conservation laws of the form,
\[ u_t + f(u)_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty) \] (1)
is based on the idea of *entropy solutions*. In this problem we will see the motivation for this notion of solution and show that entropy solutions satisfy the *Lax entropy condition* along shocks. Let \( \eta \) be a smooth convex function, we call this an *entropy solution* and show that entropy solutions satisfy the *Lax entropy condition*.

\[ q(u) := \int_0^u \eta'(v)f'(v) \, dv. \]

The pair \((\eta, q)\) is called an *entropy pair*.

(a) Show that if \( u^\varepsilon \) solves the viscous approximation to the conservation law,
\[ u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty) \] (2)
and \((\eta, q)\) is an entropy pair then,
\[ \eta(u^\varepsilon)_t + q(u^\varepsilon)_x \leq \varepsilon \eta(u^\varepsilon)_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty). \] (3)

(b) Show that if \( u^\varepsilon \) is an entropy solution of (1) in a region \( V \) of space-time and \( u \) is smooth on either side of a smooth parametrized curve (a shock) \( C = \{ (\gamma(t), t) : t \in I \subseteq \mathbb{R} \} \) with \( u, u_\ell \), and \( u_r \) uniformly continuous in the regions \( V_\ell \) and \( V_r \) to the left and right of \( C \) then the shock satisfies the Lax entropy condition,
\[ f'(u_\ell(\gamma(t), t)) \geq \gamma'(t) \geq f'(u_r(\gamma(t), t)) \quad \text{for all} \quad t \in I. \]

Here \( u_\ell \) and \( u_r \) are the left and right limits of \( u \) along \( C \) respectively.

**Hint:** First show that (4) implies a kind of Rankine-Hugoniot condition for \( \eta(u) \). Then choose a good entropy/entropy flux pair.

(c) Show that if \( u \) is an entropy solution of (1), satisfying that \( u(x, t) \) has compact support in \( x \) for each \( t > 0 \), with initial data \( u(x, 0) = u_0(x) \) then for every \( p \geq 1 \),
\[ \int_\mathbb{R} |u(x, t)|^p \, dx \leq \int_\mathbb{R} |u_0(x)|^p \, dx \quad \text{for all} \quad t > 0. \] (5)

Give an example of a weak solution of Burger’s equation for which (5) does not hold.

Problem 2. Consider the viscous approximation of a scalar conservation law,
\[ u_t + f(u)_x = \varepsilon u_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty), \] (6)
for a smooth convex flux \( f \). Carefully show that there exists a (non-trivial) travelling wave solution \( u(x, t) = v(x+ct) \) connecting two values \( u_\ell, u_r \in \mathbb{R} \),
\[ v(-\infty) = u_\ell, \quad v'(-\infty) = 0 \quad \text{and} \quad v(+\infty) = u_r, \quad v'(+\infty) = 0, \]
if and only if \( c = (f(u_\ell) - f(u_r))/(u_\ell - u_r) \) and \( f'(u_\ell) > c > f'(u_r) \).
**Hint:** There is some (basic) ODE theory involved in this problem, you can look back at the note I gave you at the beginning of the quarter or come talk to me if you have any issues.

**Problem 3.** Shearer and Levy: Chapter 13, problem 10.

**Remark:** Although I am only giving you this one “computational” problem I recommend that you do some of the other problems in Chapter 13 of Shearer and Levy to get some practice computing entropy solutions of scalar conservation laws. For example problems 6 and 9 in Chapter 13, also problem 7 although you would need to read Section 13.2.2 for that.

**Note for Problems 4 and 5:** The below problems involve the Hopf-Lax formula for the solution of a Hamilton-Jacobi equation,

\[ u_t + H(Du) = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \text{ with } u(x, 0) = g(x), \]

with \( g \) Lipschitz continuous and Hamiltonian \( H : \mathbb{R}^n \rightarrow \mathbb{R} \) satisfying

1. \( H \) convex and
2. \( \lim_{|p| \rightarrow \infty} \frac{H(p)}{|p|} = +\infty. \)

The Lagrangian \( L \) is defined as the convex dual (Legendre transform) of \( H \),

\[ L(v) = H^*(v) = \sup_{p \in \mathbb{R}^n} \{ p \cdot v - H(p) \}. \]

Then the Hopf-Lax formula gives a weak solution of the Hamilton-Jacobi equation by the formula,

\[ u(x, t) = \inf_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\}. \]

**Problem 4.** (Evans 2nd Edition, Chapter 3 problem 13) Prove that the Hopf-Lax formula reads

\[ u(x, t) = \inf_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\} = \inf_{y \in B(x, Rt)} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\} \]

for \( R = \sup_{x \in \mathbb{R}^n} |DH(Dg)|, \) \( H = L^* \). (This proves finite speed of propagation for Hamilton-Jacobi equations with convex \( H \) and Lipschitz \( g \).)

**Problem 5.** (Evans 2nd Edition, Chapter 3 problem 14) Let \( E \) be a closed subset of \( \mathbb{R}^n \). Show that if the Hopf-Lax formula could be applied to the initial value problem,

\[
\begin{cases}
    u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
    u = \begin{cases} 
        0 & x \in E \\
        +\infty & x \in \mathbb{R}^n \setminus E 
    \end{cases} & \text{on } \mathbb{R}^n \times \{t = 0\}
\end{cases}
\]

it would give the solution,

\[ u(x, t) = \frac{1}{4t} \text{dist}(x, E)^2. \]