Problem 1. Let $A : \mathbb{R}^n \to S_n$ where $S_n$ is the space of real symmetric $n \times n$ matrices. Suppose that $0 \leq A(x) \leq \Lambda$ in the sense of matrices, i.e. $\Lambda I - A$ and $A$ are non-negative definite. Show that there is at most one smooth solution of the wave type equation,

$$
\begin{cases}
  u_{tt} - \nabla \cdot (A(x) \nabla u) = f(x, t) & \text{in } \mathbb{R}^n \times (0, \infty) \\
u(x, 0) = \phi(x), \ u_t(x, 0) = \psi(x) & \text{in } \mathbb{R}^n.
\end{cases}
$$

Hint: You should show a finite speed of propagation property as we did in class. One of the key elements is to select a backwards light cone with a large enough propagation speed. Note that it is not possible to do this by looking at the energy on all of $\mathbb{R}^n$ (why?). You may find it useful to prove the following inequality for all vectors $v, w \in \mathbb{R}^n$ (which uses just that $A(x) \geq 0$),

$$
2 \langle A(x)v, w \rangle \leq \langle A(x)v, v \rangle + \langle A(x)w, w \rangle.
$$

Problem 2. Shearer and Levy: Chapter 4, Problem 6.

Problem 3. Let $u$ be the D’Alembert solution of the 1-d wave equation

$$
\begin{cases}
  u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
u(x, 0) = \phi(x), \ u_t(x, 0) = \psi(x) & \text{in } \mathbb{R}.
\end{cases}
$$

Under what condition on the initial data $\phi, \psi$ will $u$ be a travelling wave moving to the right (i.e. $u(x, t) = F(x - ct)$ for some $F$)? Under what condition will it be a travelling wave moving to the left?

Problem 4. [Evans 2nd Ed., Ch. 2 problem 23] Let $S$ denote the square lying in $\mathbb{R} \times (0, \infty)$ with corners at the points $(0, 1), (1, 2), (0, 3)$ and $(-1, 2).$ Define,

$$
f(x, t) := \begin{cases} 
-1 & \text{for } (x, t) \in S \cap \{t > x + 2\} \\
1 & \text{for } (x, t) \in S \cap \{t < x + 2\} \\
0 & \text{else}
\end{cases}
$$

Assume that $u$ solves,

$$
\begin{cases}
  u_{tt} - u_{xx} = f(x, t) & \text{in } \mathbb{R} \times (0, \infty) \\
u = 0, \ u_t = 0 & \text{on } \mathbb{R} \times \{t = 0\}.
\end{cases}
$$

Describe the shape of $u$ for times $t \geq 3.$

Hint: You should use the Duhamel’s formula for the wave equation (which I will derive in class on Tuesday) which in this case says,

$$
u(x, t) = \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y, s) \, dy \, ds = \int_{\Delta(x,t)} f(y, s) \, dy \, ds.
$$

Here $\Delta(x, t) = \{(y, s) \in \mathbb{R} \times (0, \infty) : x - (t - s) \leq y \leq x + (t - s)\}$ is the backwards light cone from $(x, t).$