Problem 1. Solve the following initial value / initial boundary value problems using the method of characteristics, you can refer to Chapter 3.1 of the textbook if you need a reference outside of your lecture notes. I usually find it helpful to draw a picture of the characteristics.

(a) Let \( f, g : [0, \infty) \to \mathbb{R} \) with \( f(0) = g(0) \), solve in terms of \( f, g \): 
\[
\begin{cases}
  u_t + x^{1/2}u_x = 0 & \text{for } (x, t) \in (0, \infty) \times (0, \infty) \\
  u(x, 0) = f(x) & \text{and } u(0, t) = g(t)
\end{cases}
\]

(b) Let \( b \in \mathbb{R}^n, c \in \mathbb{R} \) write down the general solution of, 
\[
\begin{cases}
  u_t + b \cdot \nabla_x u = cu & \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty) \\
  u(x, 0) = f(x) & x \in \mathbb{R}^n.
\end{cases}
\]

Problem 2. Use the method of characteristics to solve the following equation 
\[
\begin{cases}
  u_t + x|u_x|u_x = 0 & (x,t) \in \mathbb{R} \times (0, \infty) \\
  u(x,0) = f(x) & x \in \mathbb{R}.
\end{cases}
\]
After finding the general solution, suppose additionally that \( f(x) = 0 \) for \( |x| \leq r \). Show that there is minimal time \( T(r) \) so that for any such \( f \) the corresponding solution \( u \) satisfies \( u(\cdot,t) \equiv 0 \) for all \( t \geq T \), calculate \( T(r) \).

Problem 3. Let \( U \) any bounded open set of \( \mathbb{R}^n \) with smooth (or piecewise smooth) boundary. We call \( \nu \) to be the outward unit normal to \( \partial U \) the boundary of \( U \). Prove the following identities which come up very often in studying Laplace, heat and wave equations:
\[
\int_U |\nabla u|^2 + u\Delta u \, dx = \int_{\partial U} u \frac{\partial u}{\partial \nu} \, dS
\]
\[
\int_U u\Delta v - v\Delta u \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, dS.
\]
You can and should make use of the divergence theorem.

Problem 4. [Evans, 1st edition, Ch. 2 Problem 2] Prove that Laplace’s equation \( \Delta u = 0 \) is rotation invariant; that is, if \( O \) is an orthogonal \( n \times n \) matrix and we define
\[
v(x) := u(Ox) \quad (x \in \mathbb{R}^n),
\]
then \( \Delta v = 0 \).

Problem 5. [Evans, 1st edition, Ch. 2 Problem 4] (See Section 8.3 in Shearer and Levy for relevant material) Let \( U \) be a bounded domain of \( \mathbb{R}^n \). We say \( u \in C^2(U) \) is \textit{subharmonic} if
\[
-\Delta u \leq 0 \quad \text{in } U.
\]

(a) Prove for subharmonic \( v \) that 
\[
v(x) \leq \frac{1}{|B(x, r)|} \int_{B(x, r)} v(y) \, dy \quad \text{for all } \overline{B(x, r)} \subset U.
\]
(b) Prove that the weak maximum principle holds for subharmonic \( v \in C^2(U) \cap C(\overline{U}) \).
(c) Let \( \phi : \mathbb{R} \to \mathbb{R} \) be smooth and convex. Assume \( u \) is harmonic and \( v = \phi(u) \). Prove that \( v \) is subharmonic.
(d) Prove \( v = |Du|^2 \) is subharmonic whenever \( u \) is subharmonic (you can assume that \( |Du|^2 \) is \( C^2(U) \)).
Problem 6. [Bonus]

(a) Let \( \rho : \mathbb{R}^n \to [0, \infty) \) be a smooth radially symmetric function supported in \( \overline{B(0,1)} \) with \( \int_{\mathbb{R}^n} \rho(x) \, dx = 1 \) (the existence of such a function is not totally obvious, but suppose you have it). Consider the sequence,

\[
\rho_\varepsilon(x) = \varepsilon^{-n} \rho \left( \frac{x}{\varepsilon} \right),
\]

this sequence is called a mollifier. Suppose that \( f \in C(\mathbb{R}^n) \), show that \( (\rho_\varepsilon \ast f) \) — called a mollification of \( f \) — is smooth for every \( \varepsilon > 0 \) and \( \rho_\varepsilon \ast f \to f \) pointwise and locally uniformly as \( \varepsilon \to 0 \).

(b) Let \( U \subset \mathbb{R}^n \) open, we say that \( u \in C(U) \) is weakly harmonic if, for every \( \varphi \in C^2_c(U) \),

\[
\int_U u(x) \Delta \varphi(x) \, dx = 0.
\]

Show that if \( u \) is weakly harmonic in \( U \) then actually \( u \) is harmonic in \( U \) as well.