Initial / Boundary Value problem

\((D)\) \begin{align*}
    u_t - \Delta u &= 0 & \text{in} & \quad \Omega_T = \mathbb{U} \times (0,T) \\
    u(x,t) &= g(x,t) & \text{on} & \quad \Gamma_T = \mathbb{U}_T \cup \mathbb{U}_T \\

\end{align*}

\( \mathbb{U} \subseteq \mathbb{R}^n \) bounded domain

\( \mathbb{U}_T \) called the parabolic cylinder

\( \mathbb{U}_T \) called parabolic boundary

It is the sides and bottom of the cup

boundary data needs to be specified

on the sides and bottom of \( \mathbb{U}_T \)

(i.e., on \( \Gamma_T \))

Thus suppose there is at most one solution of \((D)\) in \( C^{2,1}(\overline{\Omega_T}) \).

proof suppose \( u_1, u_2 \) both solve \((D)\)

\( u = u_1 - u_2 \) solves

\( \begin{cases} 
    \partial_t u - \Delta u = 0 & \text{in} \quad \Omega_T \\
    u |_{\partial \Omega} = 0 & \text{on} \quad \Gamma_T 
\end{cases} \)
multiply equation by \( u \),

\[
u \frac{\partial u}{\partial t} - u \Delta u = 0 \quad \text{on} \quad \Omega_T
\]

\[
\frac{\partial}{\partial t} \left( \frac{u^2}{2} \right) - u \Delta u \geq 0 \quad \text{in} \quad \Omega_T
\]

integrate over \( \Omega \) at fixed \( t \).

\[
0 = \int_\Omega \frac{\partial}{\partial t} \left( \frac{u^2}{2} \right) - u \Delta u \, dx = \int_\Omega \frac{d}{dt} \int_\Omega u \chi_t^2 \, dx + \int_\Omega 10u^2 \, dx
\]

\[
- \int_\partial \Omega u \frac{\partial u}{\partial n} \, ds
\]

\[
= \frac{d}{dt} \int_\Omega u \chi_t^2 \, dx + \int_\Omega 10u^2 \, dx
\]

\[
\geq \frac{d}{dt} \int_\Omega u \chi_t^2 \, dx
\]

so

\[
\frac{d}{dt} \int_\Omega u \chi_t^2 \, dx \leq 0
\]

\[
\Rightarrow \int_\Omega u \chi_t^2 \, dx \leq \int_\Omega u(x,0)^2 \, dx = 0
\]

\( \exists \Omega_0 = 0 \) on \( \Omega \geq 3 \chi \times T = 03 

\Rightarrow \quad u \chi_t \chi = 0 \quad \text{in} \quad \Omega_T.
This is called the energy method for uniqueness.

That equation also satisfies a maximum principle.

**Theorem (Max principle)** Let \( u \in C^0(\bar{U}_T) \cap C^2(\Omega_T) \) solve \( u_t = \Delta u \) in \( U_T \).

Then \( \max_{\bar{U}_T} u(x, t) = \max_{\partial U_T} u(x, T) \).

i.e., maximum in parabolic domain occurs on parabolic boundary.

\[ \uparrow \square \]

Parabolic boundary is sides and bottom of the cup.

**Proof:** First suppose \( \Delta u - u_t < 0 \) in \( U_T \)

\[ u < \max_{\partial U_T} u \] at \( t = 0 \)

Then by continuity \( u(x, t) < \max_{\bar{U}_T} u \) for \( t \) sufficiently small.
let to be the first time that
\[
\max_{x \in V} u(x, t) = 0
\]
so
\[
\forall \varepsilon > 0, \exists x_0 \in V \text{ s.t. } u(x_0, t) < \varepsilon
\]

if the set being maximized over is empty we are done. If not then

\[\exists \text{ a sequence } x_n \in V \text{ and } t_n \uparrow t_0 \]
so that
\[
u(x_n, t_n) \geq \max_{x \in V} u(x, t) = 0
\]

since \( u < 0 \) on \( T \) and \( u \in C(\bar{V}) \)

so we have \((x_0, t_0) \in V_t\)
so that \( u(x_0, t_0) = 0 \)

\( u(x, t) \leq 0 \) for \( x \in \bar{V}, t \geq t_0 \)

\( u(x, t) < 0 \) for \( x \in \bar{V}, t < t_0 \)

so \( x_0 \) is a global max of \( u(\cdot, t_0) \)

so \( \Delta u(x_0, t_0) \leq 0 \)

and
\[ \partial_t u(x, t) = \lim_{h \to 0} \frac{u(x, t + h) - u(x, t - h)}{2h} \geq 0 \]

so
\[ (\partial_t u - \Delta u)(x_0, t_0) > 0 \]

which is a contradiction.

Now we need to do general case
\[
\begin{align*}
\text{Consider} & \quad \forall \quad v \in C^2(\bar{T}_T) \quad \text{s.t.} \quad v_0(x) = u(x, t_0) \quad \text{max} \quad u_{x-x} - 3 < 0 \quad \text{on} \quad \Gamma_T \\
\quad v_0 \leq 0 - \varepsilon < 0 < 0 \quad \text{on} \quad \Gamma_T \\
\text{and} & \quad \partial_t v_0 - \Delta v_0 = \partial_t u - \Delta u - 3 < 0 \quad \text{on} \quad \Gamma_T \\
\text{so our previous argument applies to get} & \\
\quad v_0 \leq 0 \quad \text{in} \quad \Omega_T \quad \text{or} \\
\quad \max u_{x-x} + \varepsilon + 2t < 0 \quad \text{in} \quad \Omega_T \\
\text{send} & \quad \varepsilon \to 0. \]
\]
Remark: You should recognize the method from Laplace equation. Prove Max principle for strict subsolutions. Then make a perturbation. For strict subsolutions via the condition of interior local max to contradict the equation.