Consider a simple random walk on $\mathbb{Z}$

$$(X^k_j)_{k \in \mathbb{N}} \quad \cdots \quad \frac{1}{2} \quad \frac{1}{2} \quad \cdots \quad \frac{1}{2} \quad \cdots \quad \frac{1}{2} \quad \cdots$$

$X_{k+1} = \begin{cases} X_k + 1 & \text{with prob } \frac{1}{2} \\ X_k - 1 & \text{w/ prob } \frac{1}{2} \end{cases}$

independent coin flips at each step.

$X_0 = 0$

for a different initial point $x \in \mathbb{Z}$.

call $X^x_n$ SRW started at $x$.

can also choose initial point $x$ randomly

from a distribution $\mathbb{P} : \mathbb{Z} \rightarrow \mathbb{R}_+$

\[ \sum_{x \in \mathbb{Z}} \mathbb{P}(x) = 1 \]

call this $(X^0_n)_{k \in \mathbb{N}}$.

What is the probability distribution of $X^0_n$ for the location of $X^0_n$?

call measure $\mu_n(X) = \mathbb{P} (X^0_n = x)$
$$u(x_{k+1}^1) = \left( x_{k+1}^1 = x \right)$$

$$= \frac{1}{2} \mathbb{P}(X_k^1 = x + 1) + \frac{1}{2} \mathbb{P}(X_k^1 = x - 1)$$

$$= u(x_{k+1}^1) + \frac{1}{2} (u(x+1, k) + u(x-1, k) - 2u(x, k))$$

or rearranging

$$u(x_{k+1}^1) - u(x, k) = \frac{1}{2} (u(x+1, k) + u(x-1, k) - 2u(x, k))$$

**discrete heat equation.**

**discrete heat equation tracks the evolution of the probability distribution function for simple random walk.**

**heat equation tracks evolution of pdf for Brownian motion.**

**Laplace equation:**

Consider again the SRW now on $\mathbb{Z}^2$

just to see how Laplace comes up

instead of another PDE operator

consistent with Laplace in 1-D?
let \( \Lambda \subset \mathbb{Z}^2 \) a connected bounded region.

\[
\frac{1}{4} \lor \frac{1}{2} \lor \frac{1}{4} \\
\frac{1}{4} \lor \frac{1}{2} \lor \frac{1}{4}
\]

Call \( \partial \Lambda \) the outer vertical boundary, 
\( \forall \mathbf{e} \in \mathbb{Z}^2 \) so that \( x \pm \mathbf{e}_1, x \pm \mathbf{e}_2 \) is \( \in \Lambda \)
and let \( g: \partial \mathbf{4} \Lambda \rightarrow \mathbb{R} \).

Look at \( U(x) = \mathbb{E}C \sum_{x \in \Lambda_2} g(X^x_{\mathbf{e} \in \Lambda}) \)

where \( \mathbf{e} \mathbf{r}(\Lambda) = \inf \{ \mathbf{e} \mathbf{r}_0 : X^x_{\mathbf{e} \in \Lambda} \} \)

Note \( \mathbf{e} \mathbf{r}(\Lambda) \) is a random variable

\[
\mathbf{e} \mathbf{r}(\Lambda) \in \partial \mathbf{4} \Lambda
\]

Define \( \Lambda \) as

\[
U(x) = \frac{1}{4} (u(x+e_1) + u(x-e_1) + u(x+e_2) + u(x-e_2))
\]

or

\[
\frac{1}{2} \mathbf{D} = \frac{1}{4} \left( (u(x+e_1) + u(x-e_1) - 2u(x)) + (u(x+e_2) + u(x-e_2) - 2u(x)) \right)
\]

\( \frac{1}{2} \mathbf{D} \) is the discrete Laplace operator on \( \mathbb{Z}^2 \).
\[ \begin{align*}
\Delta_2 u(x) &= 0 & x \in \Lambda \\
u(x) &= g(x) & x \in \partial \Omega \setminus \Lambda
\end{align*} \]

a Dirichlet problem for Laplace operator.

So solution of Dirichlet problem can be interpreted as the expected value of \( g \) at the location where Brownian motion started at \( x \) leaves domain \( \Omega \).

\textbf{References}

Finite Difference Schemes
The Heat Equation

\[
\begin{aligned}
\{ & u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
& u(x, 0) = f(x) & \text{in } \mathbb{R}^n \\
\end{aligned}
\]

- Smoothing
- Energy dissipation (Energy dissipation)
- Backwards ill-posedness
- Maximum principle

Mass Conservation if \( u \) solves heat equation \( \nabla \cdot u = 0 \)

\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^n} u \, dx = \int_{\mathbb{R}^n} u_t \, dx = \int_{\mathbb{R}^n} \Delta u \, dx = 0
\]

so

\[
\int_{\mathbb{R}^n} u \, dx = \int_{\mathbb{R}^n} u(0) \, dx
\]

For Scaling Invariance

Suppose we have \( u \) solving

\[
\begin{aligned}
& u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \\
\end{aligned}
\]

when is \( u_\lambda(x, t) = u(\lambda x, \lambda^2 t) \) a solution for all \( \lambda > 0 \)?

\[
\begin{aligned}
& u_\lambda_t = \lambda^2 u_\lambda & \text{in } \mathbb{R}^n \\
& \Delta u_\lambda = \lambda^2 \Delta u & \text{so } u_\lambda_t - \Delta u_\lambda = \lambda^2 u_\lambda - \lambda^2 \Delta u = 0
\end{aligned}
\]

so \( u_\lambda(x, t) = u(\lambda x, \lambda^2 t) \) is a solution when \( \lambda = 2 \).
Let's look for a scale invariant solution

\( \sqrt{x(t)} = \lambda \)

\( \sqrt{x(t)} = \lambda \sqrt{v(x, \lambda^2 t)} \quad \forall \lambda > 0 \)

so to preserve mass under the rescaling

\( \lambda = e^x - 1 \)

Note that if \( \lambda = e^x \)

\( \sqrt{x(t)} = e^{\lambda^2 t} \sqrt{\frac{\lambda^2}{e^{\lambda^2 t}}} \)

so \( x \) is determined just by a function

\( y = \frac{x}{\lambda} \quad v(y) = \sqrt{y} \)

\( \sqrt{\frac{x(t)}{\lambda}} = \frac{1}{\lambda^{1/2}} v\left(\frac{x}{\lambda}\right) \) again, since

A solution of heat can have \( \frac{df}{dt} u = 0 \)

need to choose \( \lambda = 1 \), but

let's just think about the requirement

\( 0 = \partial_t u - \partial_x \partial_x u = -\frac{1}{2} \frac{1}{\lambda^{1/2}} \frac{\lambda^{1/2}}{t^{3/2}} \frac{\lambda^{1/2}}{t^{3/2}} V'' - \frac{1}{2} \frac{1}{t^{3/2}} \frac{\lambda^{1/2}}{t^{3/2}} V' - \frac{1}{t^{1/2}} \frac{\lambda^{1/2}}{t^{3/2}} V' \)

\( = \frac{1}{t^{1/2}} \left( V'' + \frac{1}{2} \frac{\lambda^{1/2}}{t^{3/2}} V' + \frac{\lambda^{1/2}}{t^{3/2}} V' \right) \)
\[ v'' + \frac{1}{2} y v' + \frac{1}{2} v = 0 \]

\[ v'' + \frac{1}{2} (yv)' = 0 \]

Integrating

\[ \frac{y v'}{2} = \frac{v^2}{4} \]

\[ v' + \frac{1}{2} y v = A \]

Using an integrating factor

\[ (e^{y^{3/4} v})' = Ae^{y^{3/4}} \]

We can set \( A = 0 \)

Since we are just looking for one solution

\[ e^{y^{3/4} v(y)} = B \]

so \( v(y) = Be^{-y^{3/4}} \)

We choose \( B \) so that

\[ 1 = \int_{-\infty}^{\infty} v(y) \, dy = B \int_{-\infty}^{\infty} e^{-y^{3/4}} \, dy = B \sqrt{4\pi} \]

So

\[ v(y) = \frac{1}{\sqrt{4\pi}} e^{-y^{3/4}} \]

\[ E(x,t) = \frac{i}{\sqrt{2\pi}} v \left( \frac{x}{\sqrt{t}} \right) = \frac{i}{\sqrt{\pi \sqrt{t}}} e^{-x^2/4t} \]

The fundamental solution of heat eqn.
\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = 0 & \text{in } \mathbb{R} \times (0,\infty) \\
\frac{\partial u}{\partial x} (x, 0) = \phi(x)
\end{cases}
\]

We will see \( u(x, 0) = \delta_0 \).

**Higher dimensions**

In \( \mathbb{R}^n \),

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \sum_{i<j}^n \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i<j}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \\
&= 0
\end{align*}
\]

So \( u(x, t) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -|x|^2 / 4t \right) \).
Properties of the fundamental solution

$\overline{u}(x, t)$ solves heat equation away from $(0, 0)$

$x \neq y$, so

$\overline{u}(x - y, t)$ solves away from $(y, 0)$

\[
\overline{u}(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x - y|^2/4t} \overline{g}(y) \, dy
\]

Should be a solution as well.

Then let $g \in C^0(\mathbb{R}^n)$ be bounded and

\[
\overline{u}(x, t) = (\overline{u}(x, t) * g)(x)
\]

Then

1. $u(t) \in C^0(\mathbb{R}^n)$ for every $t > 0$
2. $u_t + \Delta u = 0$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$
3. $\lim_{(x, t) \to (x_0, 0)} u(x, t) = g(x_0)$
proof: \( \Omega(x, t) \equiv C^0 \) and all derivatives have exponential decay so differentiating under integral will be justified.

\[
u_t - \Delta u = \int_\Omega \left( \Omega(x-y, t) - \Delta \Omega(x-y, t) \right) g(y) \, dy
\]

\[= 0\]

so \( u \) solving heat eqn.

Let \( \varepsilon_0 > 0 \), since \( g \in C(\overline{\Omega^n}) \) \( E \neq 0 \)

so that \( |x - x^0| \leq \varepsilon \Rightarrow |g(x) - g(x^0)| \leq \varepsilon \)

\[
\left| g(x^n) - g(x^0) \right| = \int_{\Omega^n} \left| \Gamma(x-y, t) \right| (g(y) - g(x^0)) \, dy
\]

(by usual trick since \( \int_{\Omega^n} \Gamma(x-y, t) \, dy = 1 \))

\[
\leq \int_{\Omega^n} \left| g(y) - g(x^0) \right| \, dy
\]

now for \( |y - x^0| \leq \varepsilon \) \( |g(y) - g(x^0)| \leq \varepsilon \)

while for \( |y - x^0| > \varepsilon \) \( \Omega(x-y, t) \) will have

Whisky Horse \( \Omega \) t \( \to 0 \)
\[ \sup_{|x|, k} |g(x)| \leq \int_{|y-x| \leq \delta} \max(|x-y, t| \geq \delta) \left| g(y) - g(x^0) \right| \, dy \]

\[ + \int_{|y-x| > \delta} \max(|x-y, t| \geq \delta) \left| g(y) - g(x^0) \right| \, dy \]

\[ \leq 2 \cdot \int_{|y-x| \leq \delta} \max(|x-y, t| \geq \delta) \, dy + 2 \sup_{R^n} \int_{|y-x| > \delta} \, dy \]

\[ \leq 1 \]

\[ = \int_{|x-y| \geq \delta} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \, dy \]

Now if \( |x-x^0| \leq \frac{\delta}{2} \)

\[ \Rightarrow |y-x| \geq \frac{\delta}{2} - \frac{1}{2} |y-x^0| \]

for \( y \) in the region of integration.

\[ \leq \int_{|x-y| \geq \delta} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \, dy \]
Changing variables to

\[ z = \frac{y-x_0}{4\sigma t}, \quad dz = \frac{1}{(4\pi t)^{n/2}} \, dy \]

\[ = \frac{1}{(4\pi t)^{n/2}} \int_{\Omega} e^{-|z|^2} \, dz \]

\[ = \left(\frac{4\pi t}{n}\right)^{n/2} \int_{121z^2 > \frac{t}{48\sigma}} e^{-|z|^2} \, dz \to 0 \quad \text{as} \quad t \to 0 \]

since \( e^{-|z|^2} \) is integrable.

This is a (rigorous) justification that \( \Phi(t) \to 0 \) as \( t \to 0 \) in the sense of distributions.
Ockam’s Principle for the Inhomogeneous Heat Eq

\[ \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \end{cases} \]

We can think of \( f \)

We will build solution again using \( E \)

Note \( E(x-y, t-s) \) solves heat

\[ u(x,t) = \int_{\mathbb{R}^n} E(x-y, t-s) g(y) \, dy \]

\[ u(x,s) \] \text{ satisifies}

\[ \begin{cases} \frac{\partial u}{\partial t} - \Delta u (x,t) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ u(x,s) = f(x,s) & \text{in } \mathbb{R}^n \end{cases} \]
so we already know a solution

\[ u(x,t; s) = \int_{\Omega^n} \mathcal{E}(x-y, t-s) f(y, s) \, dy \, ds \]

The Duhham's principle says

\[ u(x,t; t) = \int_{\Omega^n} \mathcal{E}(x-y, t) g(y) \, dy + \int_0^t \int_{\Omega^n} \mathcal{E}(x-y, t-s) f(y, s) \, dy \, ds \]

Then, let \( f \in C^{2,1}(\Omega^n \times (0,\infty)) \) and \( g \in C^0(\Omega^n) \) hold then

\[ \text{U(x;t) from Duhham's formula solve} \]

\[ \begin{cases} u_t - Au = f(x,t) & \text{in } \Omega^n \times (0,\infty) \\ u(x,0) = g(x) & \text{in } \Omega^n \end{cases} \]

(1) \( u \in C^{2,1}(\Omega^n \times (0,\infty)) \)

(2) \( \lim_{(x,t) \to (x,0)} u(x,t) = g(x) \)
Proof. We just need to analyse

\[ u(x, t) = \int_0^t u(x, t; s) \, ds \]

Check that it is regular and solve

\[
\begin{aligned}
  & v_t - \Delta v = f(x, t) & \quad & \text{in } \mathbb{R}^n \times (0, \infty) \\
  & v(x, 0) = 0 & \quad & \text{in } \mathbb{R}^n
\end{aligned}
\]

Note: by previous result, \( u(x, t; s) \) is

\[ u(x, t; s) \to \frac{f(x, s)}{t-s} \quad \text{as} \quad t \to s \]

\[ u_t(x, t) = u(x, t; t) + \int_0^t u_t(x, t; s) \, ds \]

\[ = f(x, t) + \int_0^t u_t(x, t; s) \, ds \]

\[ \Delta u(x, t) = \int_0^t \Delta u(x, t; s) \, ds \]

\[ v_t - \Delta v = f(x, t) + \int_0^t (u_t(x, t; s) - \Delta u(x, t; s)) \, ds \]

\[ = f(x, t) \]
Initial / Boundary Value Problem

\[(D) \begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T = \Omega \times [0, T] \\ u(x, t) = g(x, t) & \text{on } \Gamma_T = \Gamma_T \cup \Gamma_T \end{cases}\]

\(\Omega_T\) called the parabolic cylinder

\(\Gamma_T\) called parabolic boundary

It is the sides and bottom of the cup

boundary data needs to be specified

on the sides and bottom of \(\Omega_T\)

\(\Gamma_T\)

Then \(\text{Suppose } \text{there is at most one solution}

of \((D)\) in } C^{2,1}(\overline{\Omega_T}) \text{ ACCURATE.}

proof