Hamilton-Jacobi Equations

(1) \[
\begin{cases}
  u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
  u(x,0) = g(x) & x \in \mathbb{R}^n
\end{cases}
\]

$H: \mathbb{R}^n \to \mathbb{R}$ called the Hamiltonian

Our goal or with the conservation laws is to ensure the solution past the time when characteristics cross. To do so it will be necessary, as before, to select the correct weak solution by using some physical principle.

We will find that (1) PDE is associated with an optimal control problem.

Characteristics for (1) are

\[
\begin{cases}
  x = D_p H(p,x) \\
  p = -D_x H(p,x)
\end{cases}
\]

Called Hamilton's ODEs

These ODE arise in classical mechanics from a variational principle.
Let $L: \mathbb{R}^n \to \mathbb{R}$ a given smooth function. Consider the action

$$I[w] = \int_0^T L(w(s), \dot{w}(s)) \, ds$$

for $x, y \in \mathbb{R}^n$ and $t > 0$. The cost functional is defined.

We wish to find a minimal cost/action path from $y$ to $x$. We wish to find

$$I[x(\cdot)] = \min_{w(\cdot)} I[w(\cdot)].$$

Let us assume we can find such a path $x(\cdot)$. 

for $x, y \in \mathbb{R}^n$ and $t > 0$. The action

$$I[w] = \int_0^T L(w(s), \dot{w}(s)) \, ds$$

and $t > 0$. The action

$$I[w] = \int_0^T L(w(s), \dot{w}(s)) \, ds$$

We define for

$$w : [0, 1] \to C([0, T], \mathbb{R}^n); w(0) = y, w(T) = x$$

smooth paths from $y$ to $x$.
Then (Euler-Lagrange Eqs)  \( P \) is the form of \( x \) solves

\[
- \frac{d}{ds} \left( DfL(x, u) + DL(x, u) \right) = 0 \quad \text{as } t.
\]

Proof: as we did for previous variational principle, we look at the "directional derivative"

\[
\theta(t) = \theta(x + \epsilon x_A) \quad \text{for } \epsilon > 0
\]

let \( u \) smooth \( [0,T] \to \mathbb{R}^n \)

\[
\theta(0) = \theta(T) = 0 \quad \text{so that}
\]

\[
w = x + \epsilon x_A
\]

\[
\theta(t) = \theta(w) \quad \forall t \geq 0.
\]

so \( \theta(0) = I \{ x + \epsilon x_A \} \) has a min at \( \epsilon = 0 \)

\[
\Rightarrow \quad \theta'(0) = 0 \quad \text{if } \text{it exists}\).
\]

\[
i(t) = \int_0^t \left( DfL(x + \epsilon x_A, x + \epsilon x_A) + DL(x + \epsilon x_A, x + \epsilon x_A) \right) ds
\]

\[
\Theta'(0) = \int_0^t DfL(x + \epsilon x_A, x + \epsilon x_A) ds + DL(x + \epsilon x_A, x + \epsilon x_A) \theta ds
\]

setting \( \epsilon = 0 \)
\[ 0 = \delta'(t_0) = \int_0^{t_0} \left( \frac{d}{ds} L(x, t) \right) \psi \, ds \]

and integrate by parts using

\[ \psi(0) = \psi(t_0) = 0 \]

\[ = \int_0^{t_0} \left( -\frac{d}{ds} \left( \frac{d}{dt} L(x, t) \right) + \frac{d}{dt} L(x, t) \right) \psi \, ds \]

Since this is zero for all \( \psi \in C^2(\mathbb{R}; \mathbb{R}) \)

we obtain

\[ \int \left( -\frac{d}{ds} \left( \frac{d}{dt} L(x, t) \right) + \frac{d}{dt} L(x, t) \right) \psi \, ds = 0 \]

for \( t \in \mathbb{R} \)

\[ \text{Example 1: } L(x, t) = \frac{1}{2} m q \dot{x}^2 - \phi(x) \]

Electric energy potential energy

\( \ddot{x} \) eqn is Newton's law

\[ m \ddot{x} = -\frac{d}{dx} \phi(x) \]

\[ L(q, x) = \begin{cases} 0 & \text{if } |q| < 1 \\ \infty & \text{if } |q| \geq 1 \end{cases} \]

Cost functional is minimal time to move from \( y \) to \( x \)

\( v/ \) speed \( \leq 1 \). E-L eqn is harder to interpret.
A payoff problem

\[ U(x, t) = \inf \left\{ \int_0^t L(w(s), w_t(s)) \, ds + g(w(t)) \mid w(0) = y, \, w(t) = x \right\} \]

Optimal trajectory solves the Hamilton-Jacobi equation

\[ \frac{d}{ds} (\partial_t U(x,s)) = 0 \quad 0 \leq s \leq t \]

Dynamic programming principle \( \forall \quad 0 < t \leq T \)

\[ U(x, t) = \inf \left\{ \int_0^t L(w(s), w_t(s)) \, ds + u(y, z) \mid w(s) = y, \, w(t) = x \right\} \]

Using the above infinitesimally

\[ U(x, t+h) = \inf \left\{ \int_t^{t+h} L(w(t), w_t(t)) \, dt + u(y, z) \mid w(t) = y, \, w(t+h) = x \right\} \]

\[ = \inf_{u} \left\{ h L(v(x), x) + u(x, z, x, t) \right\} + O(h^2) \]

\[ = \inf_{v} \left\{ h L(v(x), x) + u(x, t) + h v \cdot Du(x, t) \right\} + O(h^2) \]

\[ = u(x, t) + \inf \left\{ -v \cdot Du(x, t) + L(v(x), x) \right\} + O(h^2) \]
\[ u_t = \inf_{v} \left\{ -u \cdot Dv(x,t) + L(v,x) \right\} \]

\[ u_t + \sup_{v} \left\{ v \cdot Dv(x,t) - L(v,x) \right\} = 0 \]

\[ = H(Dv(x)) \]

---

**Proof of DPP:**

\[ u(x,t) = \inf \left\{ \int_0^t L(w(s),w) \, ds + g(y) \mid w(0) = y, \quad w(t) = x \right\} \]

Let \( y \in \mathbb{R}^n \), take \( u \) optimal for \( u(y,s) \), i.e.

\[ u(y,s) = \int_0^t L(w(s),w) \, ds + g(w(s)) \]

and let

\[ \overline{w}(t) = \begin{cases} w(s) & 0 \leq s \leq t \\ w \phi(\tau) & 0 \leq \tau \leq s \end{cases} \]

with \( w \) any path s.t. \( w(0) = y, \quad w(t) = x \)

Then

\[ u(x,t) \leq \int_0^t L(\overline{w},\overline{w}) \, ds + g(\overline{w}(0)) \]

\[ = u(y,s) + \int_s^t L(\overline{w}(\tau),\overline{w}(\tau)) \, d\tau \]

Taking \( u \) over \( w \) yields

\[ u(x,t) \leq \inf \left\{ \int_s^t L(\overline{w}(\tau),\overline{w}(\tau)) \, d\tau + u(y,s) \mid w(0) = y, \quad w(t) = x \right\} \]
for the other direction let $u_*$ optimal

for $u_{x_1, t} \geq \int_0^t L(w_\xi, w_\xi) \, dt + g(w_{x_1(0)})$

$$= \int_s^t L(w_\xi, w_\xi) \, dt + \int_s^t L(w_\xi, w_\xi) \, dt + g(w_{x_1(0)})$$

$$\geq \int_s^t L(w_\xi, w_\xi) \, dt + u(c(w_\xi(s), s))$$

$$\Rightarrow u(x_1, t) \geq \inf \left\{ \int_s^t L(w, w) \, dt + u(y, s) :\right\} \right| \begin{array}{l}
\text{w}(t) = y, \ w(t) \leq x
\end{array}$$
Suppose that \( x \) is an action minimizer (or critical point) call

\[ p(s) = \text{D}_x L(x(s), v(s)) \quad \forall s \in I \]

\( p(-1) \) is called the generalized momentum. 

\( x \) - position, \( \dot{x} \) - velocity.

Now we suppose that \( x, p \in \mathbb{R}^n \)

\[ p = \text{D}_v L(v, x) \] can be uniquely solved for \( v \) as a smooth fan of \( p, x \)

\[ v = v(p, x) \]

**Def.** The Hamiltonian associated with the Lagrangian \( L \) is

\[ H(p, x) = p \cdot v(p, x) - L(v(p, x), x) \]
Legendre Transform

Now we put some additional assumptions:

1. \( \forall y \, L(y) \) is convex

2. \( \lim_{y \to \infty} \frac{L(y)}{|y|} = +\infty \) (coercivity)

Definition: The Legendre transform of \( L \) is

\[
L^*(p) = \sup_{q \in \mathbb{R}^n} \left\{ p \cdot q - L(q) \right\} \quad (p \in \mathbb{R}^n)
\]

Suppose the supremum is achieved by \( q^* \in \mathbb{R}^n \)

\[
L^*(p) = p \cdot q^* - L(q^*)
\]

and

\[
q^* \mapsto p \cdot q - L(q)
\]

has maximum at \( q^* \)

Then \( p = DL(q^*) \) is solvable for \( q \)

(\( q^* \) unique, although not necessary)

Uniquely:

\[
L^*(p) = p \cdot q^* - L(q^*) = p \cdot q^* - p \cdot q^*(p^*)
\]

\[
L^*(p) = p \cdot q^* - L(q^*)
\]
which is the Hamiltonian $H$ associated with $L$.

We thus call $H = L^*$.

Now conversely given a Hamiltonian $H$, can we find $L$?

This (convex duality) Assume $L$ convex, coercive and define $H = L^*$ then

1. $H$ is convex and coercive

2. $H^* = L$

Proof: We call $H$ and $L$ convex dual functions

$p \rightarrow p \cdot q - L(q)$ is linear

$H(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(q) \}$ is convex.

$H(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(q) \}$

Take $q = \frac{R p}{|p|}$

$H(p) = \sup_{q \in \mathbb{B}(0, R)} \{ p \cdot q - L(q) \}

\geq R |p| - \max_{q \in \mathbb{B}(0, R)} L(q)

\geq R |p| - \max_{q \in \mathbb{B}(0, R)} L(q)

\geq R |p|

\Rightarrow \lim_{|p| \to \infty} \frac{|H(p)|}{|p|} \geq R$ for every $R > 0$.
\( \forall p, q \)

\[
H(p) + L(q) \geq p \cdot q
\]

so

\[
L(q) \leq \sup_{p \in \mathbb{R^n}} \{ p \cdot q - H(p) \} = H^*(q)
\]

and

\[
H^*(q) = \sup_{p \in \mathbb{R^n}} \{ p \cdot q - \sup_{r \in \mathbb{R}^n} \{ r \cdot q - L(r) \} \}
\]

\[
= \sup_{p \in \mathbb{R^n}} \inf_{r \in \mathbb{R}^n} \{ p \cdot (q - r) + L(r) \}
\]

Since \( L \) convex \( \Rightarrow \) it has a supporting hyperplane at \( q \)

\[
L(r) \geq L(q) + s \cdot (r - q) \quad \forall r \in \mathbb{R}^n
\]

Thus

\[
\inf_{r \in \mathbb{R}^n} \{ s \cdot (q - r) + L(r) \}
\]

\[
\leq L(q) \quad \text{from choice of } s.
\]
The Hopf-Lax formula defines

\[ u(x,t) = \inf \left\{ \int_0^t L(w(s)) \, ds + g(y) \mid w(0) = y, w(t) = x \right\} \]

will hold at \( t \) value (in a sense),

\[ \begin{cases} 
  u_t + H(Du) = 0 & \text{in } (\mathbb{R}^n \times (0,\infty)) \\
  u(x,0) = g(x) 
\end{cases} \]

where \( H = L^x \) or \( L^t \) depending on our starting point.

The (Hopf-Lax formula)

but \( x \in \mathbb{R}^n \), too. Then \( u \) as in \( c_1 \) is

\[ u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t L \left( \frac{x-y}{t} \right) + g(y) \right\} \]

(i.e. optimal trajectories in \( c_1 \) are straight lines)
proof of Hopf-Lax formula:

one direction is easy

take \( u(s) = y + \frac{s}{t} (x-y) \) so set

\[
\begin{align*}
\tilde{u}(t) &\leq \int_0^t L(\tilde{u}(s)) \, ds + g(y) \\
&= \int_0^t L\left(\frac{x-y}{t}\right) \, ds + g(y) \\
&= tL\left(\frac{x-y}{t}\right) + g(y)
\end{align*}
\]

take inf over \( y \in \mathbb{R}^n \) to obtain

\[ u(x,t) \leq \inf_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \]

if \( u \) is any \( C^1 \) for \( u(t) = x \) then

calling \( y = w(0) \)

\[
L\left( \int_{t/2}^{t} L(\tilde{w}(s)) \, ds \right) \leq \frac{1}{t} \int_0^t L(\tilde{w}(s)) \, ds \quad \text{(Fron)}
\]

so

\[
\begin{align*}
\tilde{u}(t) &> \inf_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \\
&\geq \inf_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}
\end{align*}
\]

(\text{Inf}) super-linear growth of \( L \) to show \( \inf \) is achieved.
For each \( x \in \mathbb{R}^n \), \( 0 \leq s < t \) we have:

\[
U(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s) L \left( \frac{x-y}{t-s} \right) + u(y,s) \right\}
\]

Proof:

1. Fix \( y \in \mathbb{R}^n \), \( 0 < s < t \) and choose \( z \) so that

\[
y = \frac{y+y+z}{2}
\]

Next,

\[U(y,s) = g(t-s) \frac{y-x}{2} + g(t-s)
\]

Since \( L \) is convex,

\[
L \left( \frac{x-2y}{t} \right) \leq (1 - \frac{s}{t}) L \left( \frac{x-y}{t-s} \right) + \frac{s}{t} L \left( \frac{y-x}{s} \right)
\]

so

\[
U(x+t) \leq t L \left( \frac{x-2y}{t} \right) + g(t) \leq (t-s) L \left( \frac{x-y}{t-s} \right) + s L \left( \frac{y-x}{s} \right) + g(t-s)
\]

for each \( y \in \mathbb{R}^n \).

Now choose \( w \) s.t.

\[
u(x,t) = t L \left( \frac{x-w}{t} \right) + g(t-s)
\]

where

\[y = \frac{z}{t} x + \left(1 - \frac{s}{t}\right) w
\]
**Lemma** (Lipschitz estimate) The function $u$ is Lipschitz continuous in $\mathbb{R}^n \times (0, \infty)$ and $u = g$ on $\mathbb{R}^n \times \{t = 0\}$.

**Proof:** Fix $b > 0$, $x, \hat{x} \in \mathbb{R}^n$. Let $y \in \mathbb{R}^n$ s.t.

$$b L \left( \frac{x-y}{t} \right) + g(y) = u(x,y)$$

Then

$$u(x,t) - u(x_0,t) \geq \inf_{z \in \mathbb{R}^n} \left\{ t L \left( \frac{\hat{x}-z}{t} \right) + g(z) \right\} - t L \left( \frac{x-y}{t} \right) - g(y) \geq \text{Lip}(g) |\hat{x} - x|$$

Interchanging roles of $\hat{x}, x \Rightarrow u(x_0,t) - u(x,y) \geq \text{Lip}(g) |\hat{x} - x|$

Now for time continuity, we just prove at $t = 0$.

Let $t > 0$. Choosing $y = x$ in Hopf-Lax

$$u(x,t) \leq t L(0) + g(x)$$

and

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t L \left( \frac{x-y}{t} \right) + g(y) \right\}$$

$$\geq g(x) + \max_{y \in \mathbb{R}^n} \left\{ -\text{Lip}(g)|x-y| + t L \left( \frac{x-y}{t} \right) \right\}$$

$$\left( z = \frac{x-y}{t} \right) = g(x) - t \max_{2 \in \mathbb{R}^n} \left\{ \text{Lip}(g) |z| - L(z) \right\}$$

$$\geq g(x) - t \max_{w \in \mathbb{R}} \text{Lip}(g) \max_{z \in \mathbb{R}} \left\{ |wz| - L(z) \right\}$$
\[ u(x,t) = \max_{y \in \mathbb{R}^n} h(y) \]

so that

\[ |u(x,t) - g(x)| \leq C t + b / \]

\[ C = \max \left( 1L(y), \max_{\text{poly}Lip(y)} \right) \]

Using dynamic programming principle and Lipschitz continuity in span of \( u(x,t) \), it yields time Lipschitz continuity.

Now by Redemacher's fan Lipschitz \( u \) is

fins. are differentiable a.e.

Then let \( x \in \mathbb{R}^n \), \( t > 0 \) and \( u \) defined by Hopf-Lax formula differentiable at \( (x,t) \). Then

\[ u_t \text{ at } (x,t) + \text{ Hessian of } \text{ at } (x,t) = 0 \]

**Proof:** Fix \( q \in \mathbb{R}^n \), \( h \geq 0 \), by DPP

\[ u(x + hq, t + h) = \min_{y \in \mathbb{R}^n} \left\{ hL \left( \frac{x + h(q+y)}{h} \right) + u(y,t) \right\} \]
subsolution: \[ u(x+th, t+h) = \min_{y_0} \left\{ \frac{h}{2} L \left( \frac{x+th-y}{h} \right) + u(y, t) \right\} \]

\[ C_y \geq x \]

\[ h L \left( \frac{x + th - y}{h} \right) + u(y, t) \]

so \[ \frac{u(x+th, t+h) - u(x, t)}{h} \leq L \left( \frac{x + th - y}{h} \right) \]

since \[ u \] is differentiable at \( x, t \)

\[ g_1 \cdot D_u (x, t) + u_t (x, t) \leq L \left( \frac{x + th - y}{h} \right) \]

valid for all \( g \) so

\[ u_t (x, t) + \max_{q \in \Omega} \left\{ g_1 \cdot D_u (x, t) - L \left( \frac{x + th - y}{h} \right) \right\} \leq 0 \]

\[ H (D_u (x, t)) \]

so \[ u_t (x, t) + H (D_u (x, t)) \leq 0 \]

supersolution: choose \( \xi \) s.t.

\[ u(x, t) = \frac{t}{2} L \left( \frac{x-\xi}{t} \right) + g(\xi) \]

for \( L > 0 \) and \( \xi \) s.t. \( x < t-\xi \), \( y = \frac{x}{t} + \frac{1-\frac{z}{t}}{s} \)

then \[ \frac{x-\xi}{t} = \frac{y-\xi}{s} \] and

\[ u(x, t) - u(y, s) \geq \frac{t}{2} L \left( \frac{x-\xi}{t} \right) + g(\xi) - \left[ s L \left( \frac{y-\xi}{s} \right) + g(\xi) \right] \]
\[ u(x, t) - u(y, t) > (t-s) L \left( \frac{x-y}{t} \right) \]

\[ u(x, t) - u(x - \frac{h}{4}(x-z), t-h) \geq L \left( \frac{x-z}{t} \right) \]

Letting \( h \to 0 \)

\[ u(x, t) \geq \frac{x-z}{t} \cdot Du(x, t) + u_t(x, t) \geq L \left( \frac{x-z}{t} \right) \]

and therefore

\[ u_t(x, t) + H(1, 0) \geq u_t(x, t) + \max_{v \in \partial \Omega} \{ v \cdot Du(x, t) - L(v) \} \]

\[ \geq u_t(x, t) + \frac{x-t}{t} \cdot Du(x, t) - L \left( \frac{x-z}{t} \right) \]

\[ \geq 0 \]

Thus the function \( u \) given by (26) is Lipschitz continuous, differentiable a.e. in \( \mathbb{R}^n \times (0, \infty) \) and solves the IVP

\[ \begin{cases} 
  u_t + H(1, 0)u = 0 & \text{a.e. in } \mathbb{R}^n \times (0, \infty) \\
  u = g & \text{on } \mathbb{R}^n \times \{ t = 0 \} 
\end{cases} \]
\[ \begin{cases} \frac{\partial u}{\partial t} + \frac{4}{3} (\ln x)^2 = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
0 & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \]

\[ u_1(x, t) = 0 \]

also \[ u_2(x, t) = \begin{cases} \frac{x - t}{x + t} & -2 \leq x \leq 0 \\
x + t & x > 0 \\
x - t & x < 0 \end{cases} \]

Only one solution a.e.

doesn't exist not differentiable

\[ x = 0, \pm t \]

\[ \phi \]

\[ |\phi'| > 1 \]

a.c.

and \[ \phi \geq 0 \]

\[ u_1 |_{\partial \Omega} = \begin{cases} \phi'(x - t) & \text{set: } \phi' \downarrow \end{cases} \]

\[ u_1 |_{\partial \Omega} = 0 \]

and \[ \partial_t u + H(Du) = 0 \text{ a.e. in } \mathbb{R}^n \times (0, \infty) \]

\[ \text{Any thing like } \]

\[ |\phi'|^2 \leq 1 \]

\[ \text{set: } \phi' \uparrow \]
Some Examples of HJ eqns

Level set evolutions:

We are interested in making a formulation of

 hypersurface $\Gamma_t$


 boundary of common $\Gamma_t$

 outer normal $n_\Gamma = \partial B(0,1)$

want to solve a the PDE, $\Gamma_t$ moves by normal velocity

$$V_n = c(x,t)$$

How to interpret this PDE?

$$\Delta u_{t} = u \frac{\partial u}{\partial \xi}$$

$\xi \in \mathbb{R}^n \times \mathbb{R}$, outer normal

$$V(x,t) = (n_t(x), m(x,t)^2)$$

$m$ is the outward normal velocity of $\Gamma_t$
Another interpretation:

\[ x_t \text{ any path lying in } \Omega_t \text{ through } x_t = x_0 \]

then \( \vec{X}_{t_0} \cdot n_{t_0} (x_0) \) is the outward normal velocity of \( \Omega_t \) at \( x_0 \).

The problem with this sort of solution too

[Diagram of a set \( \Omega \), \( \nabla_n = -1 \)]

topological changes are difficult to deal with.

The idea is to view \( \Omega_t \) as the zero level set of a function \( u(x, t) : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R} \),

\[ \Omega_t = \{ x \in \mathbb{R}^n : u(x, t) = 0 \} \]
topological changes are easy.

graph of \( U(x, t) \).

How is outward normal velocity related to \( \Omega \)?

\[ S, \quad \text{path in } \Omega \quad \frac{\partial t}{\partial t} = x \]

\[ 0 = \frac{\partial}{\partial t} U(x_i, t) = u_t + \frac{\partial}{\partial x_i} \cdot Du(x_i, t) \]

\[ = u_t (x_i, t) + \nabla u \cdot Du(x_i, t) \]

\[ = u_t (x_i, t) + \nabla u \cdot \frac{Du}{|Du|} |Du| (x_i, t) \]

(as long as \( u \neq 0 \))

\[ \text{in } \Omega \quad u \neq 0 \quad |Du| \neq 0 \]

\[ \frac{\partial u}{\partial n} = n_u (x_i) \]

so

\[ \frac{u_t}{|Du|} (x_i, t) = -V_n \]

so

\[ V_n = c(x(t)) \quad \rightarrow \quad -\frac{u_t}{|Du|} = c(x(t)) \]

Hence

\[ u_t = +c(x(t)) |Du| = 0 \]
Example: \( \mathbf{u}_t + \text{div} \mathbf{u} = 0 \quad \mathbf{w} / \quad \mathbf{u} \mid_{\partial D} = \mathbf{0} \)

\text{Hopt-Lax (formal solution)}

\[ L[\mathbf{u}] = H^*(\mathbf{u}) = \sup_{\mathbf{p} \in \Pi^n} \left\{ \mathbf{u} \cdot \mathbf{p} - H(\mathbf{p}) \right\} = \sup_{\mathbf{p} \in \Pi^n} \left\{ \mathbf{u} \cdot \mathbf{p} - |\mathbf{p}| \right\} \]

\[ = \begin{cases} 
0 & |\mathbf{u}| \leq 1 \\
+\infty & |\mathbf{u}| > 1
\end{cases} \]

so the associated variational problem is

\[ \mathbf{u}_{x|x} = \inf \left\{ g(\mathbf{y}) : \mathbf{w} \text{ smooth, } \mathbf{w} \mid_{|\mathbf{w}| \leq 1, \mathbf{w}(0) = \mathbf{y}, \mathbf{w}(T) = \mathbf{x} \right\} \]

\text{Hopt-Lax}

\[ \mathbf{u}_{x|x} = \inf_{\mathbf{y} \in \Pi^n} \left\{ tL \left( \frac{\mathbf{x} - \mathbf{y}}{t} \right) + g(\mathbf{y}) \right\} = \inf_{1y - x \leq 1} g(\mathbf{y}) \]

\[ \sup \text{sign} \text{dist} (x, \mathbf{p}_0) = \left\{ \begin{array}{ll}
\text{dist} (x, \mathbf{p}_0) & x \in \mathbf{p}_0^° \\
-\text{dist} (x, \mathbf{p}_0) & x \not\in \mathbf{p}_0^°
\end{array} \right. \]

then \( \mathbf{u}(x, t) = \inf \text{sign} \text{dist} \left( \mathbf{x}, \mathbf{p}_0 \right) \)

Claim: \( \mathbf{u}(x, t) = \left( \text{sign} \text{dist} (x, \mathbf{p}_0) - t \right) v \left( \min_{y \in \mathbf{p}_0} g(y) \right) \)
proof of Claim

clearly \[ u_{N+1} = \inf_{y \in \mathbb{Z}} g(y) \geq \inf_{x \in \mathbb{Z}} g \]

and if \( x^* = \text{argmin}_x g \) satisfies \( u_{x^*} \leq t \)

then \( u_{x^*+1} = \min_{y \in \mathbb{Z}} g \]

Call \( E = \{ x \in \mathbb{Z} : g(x) = \min_{y \in \mathbb{Z}} g \} \)

otherwise

Note that \( \text{sign dist} (x, E) = \text{dist} (x, E) - \min_{y \in E} g \)

\[ u_{x^*+1} = \min_{y \in \mathbb{Z}} \text{sign dist} (x, y) \geq \min_{y \in \mathbb{Z}} \text{dist} (x, E) - \min_{y \in E} g \]

\[ \geq \left( \text{dist} (x, E) - t \right) \vee 0 + \min_{y \in E} g \]
Viscosity Solutions of H-J Eqns

We need to make some definitions. First

\[ D_u(x) = \{ p \in \mathbb{R}^n : \text{such that } u(y) \leq u(x) + p \cdot (y-x) + o(|y-x|) \text{ as } y \to x \} \]

Superdifferential

\[ D_u(x) = \{ p \in \mathbb{R}^n : u(y) \geq u(x) + p \cdot (y-x) + o(|y-x|) \text{ as } y \to x \} \]

Subdifferential

From the proof of Hopf–Lax solution, we have

\[ u_t + H(Du) = 0 \]

where \( u \) is differentiable.

We actually could have shown

\[ \partial_+ D_u(x) \Rightarrow u_t + H(p) \leq 0 \quad \text{"subsolution"} \]

\[ \partial_- D_u(x) \Rightarrow u_t + H(p) \geq 0 \quad \text{"supersolution"} \]

Lemma (property)

(i) \( u \) is differentiable at \( x \),

\[ \iff D_u(x) = D_u(x^1) = \{ Du(x) \} \]

(ii) \( D_u(x), D_u(x^1) \) both convex sets.
(iii) if $D_+ u(x_1)$ non-empty

$\Rightarrow$ either $D_+ u(x_1) = D_- u(x_1) = \{ D u(x_1) \}$

or $D u(x_1) = \emptyset$

We say

Statement on a bit easier for non-time dependent

$(h-J)$ \[ H(C D u) = 0 \text{ in } V \]

Def $u$ is a viscosity **subsolution** of $(h-J)$ if $u$ is continuous and

(ii) **subsolution**, $\forall \ p \in D u(x_1)$

$H(p) \leq 0$

(iii) **supersolution**, $\forall \ p \in D u(x_1)$

$H(p) \geq 0$

**Remark**: This is in a sense the $L^0$ version of the weak solution condition we saw before.
**Rm2:** Although it is not immediately obvious from this definition, the definition of a viscosity solution essentially says that for every test function $u$ satisfies a "local comparison principle" with respect to $C^1$ test functions.

**Def2** $u$ is a viscosity subsolution of (HFS) if

for every $\bar{f} \in C^1_{\text{loc}}(\mathbb{R}^n \times (0, \infty))$, $(x,t) \in \mathbb{R}^n \times (0, \infty)$, every parabolic cylinder $Q_\tau(x,t) = \{(y,s) : |y-x| < \tau, 0 < s < t\}$, and any $\phi \in C^1(\mathbb{R}^n \times (0, t))$ such that

(i) if $\phi$ touches $u$ from below at $(x,t)$ in $Q_\tau(x,t)$, then

$\phi_t + H(D\phi) < 0$,

(ii) if $\phi$ touches $u$ from above at $(x,t)$ in $Q_\tau(x,t)$, then

$\phi_t + H(D\phi) \leq 0$. 

\[ \begin{align*}
\text{example:} & \quad \begin{cases} 
D_u = I \quad \text{in} \quad \Omega_1 \\
\text{u} = 0 \quad \text{on} \quad \partial \Omega_1 
\end{cases} \\
& \quad \text{u} \text{ is } \frac{1}{2} - 1x - \frac{1}{2} \\
& \quad \text{satisfies} \\
& \quad \mathcal{U} \text{ is } \{ 0 \} \setminus \{ \frac{1}{2} \} \\
& \quad \text{at} \quad x = \frac{1}{2} \\
& \quad D_{u}(\frac{1}{2}) = [-1,1] \\
& \quad D_{u}(\frac{1}{2}) = \emptyset \\
& \quad A \quad \text{pe } [-1,1] \\
& \quad |p| - 1 \leq 0 \quad \checkmark \quad \text{subsoln condition}
\end{align*} \]

\[ \begin{tikzpicture}
\draw[->] (-1,0) -- (1,0);
\draw[->] (0,-1) -- (0,1);
\end{tikzpicture} \]

\textbf{Thm:} \text{if } u \text{ viscosity subsolution, } v \text{ viscosity supersolution of } (H_3) \text{ in } \mathbb{R}^n \times (0, \infty) \text{ w.r.t. } \\
\text{u}(x,0) \leq v(x,0) \text{ then} \\
\text{u}(x,t) \leq v(x,t) \quad \forall \quad t \geq 0.\]