\[ \begin{align*}
(1) \quad & \begin{cases}
    u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
    u(x,0) = g(x) & x \in \mathbb{R}^n
\end{cases} \\
& H : \mathbb{R}^n \to \mathbb{R}
\end{align*} \]

our goal is to continue the solution past the time when characteristics cross. To do so it will be necessary, as before, to select the correct weak solution by using some physical principle.

We will find that (1) is associated with an optimal control problem.

Characteristics for (1) are

\[ \begin{align*}
\dot{x} &= DpH(p,x) \\
\dot{p} &= -DxH(p,x)
\end{align*} \]

called Hamilton's ODEs.

These ODEs arise in classical mechanics from a variational principle.
Let \( L : \mathbb{R}^n \rightarrow \mathbb{R} \) be a given smooth function called the log-arc. We write \( L(y, x) \) or \( \text{Running Cost} \).

Fix \( x, y \in \mathbb{R}^n \) and \( t > 0 \) the action or cost functional is defined as:

\[
D[w] = \int_0^t L(w(s), \dot{w}(s)) \, ds
\]

for \( w \in C^1[0, t] \cap \mathbb{R}^n \) with \( w(0) = y \) and \( w(t) = x \).

We wish to find the minimal cost/success path from \( y \) to \( x \):

\[
D[x(t)] = \min_{w \in A} D[w(t)]
\]

Let us assume we can find such a path, \( x(t) \).
The (Euler-Lagrange Eqns) for \( x \) solve

\[
\frac{d}{ds} \left( D_y L(x, x) \right) + DL(x, x) = 0 \quad 0 \leq s \leq t.
\]

Proof: as we did for previous variational principle we look at the "directional derivative"

\[
\delta I = \int_0^t \delta x \cdot \frac{d}{ds} L(x, x) ds + \frac{d}{ds} \left[ DL(x, x) \delta x \right] ds.
\]

Let \( \nu \) smooth \( \delta \mathbb{R} \to \mathbb{R}^n \),

\[
\nu(0) = \nu(t) = 0 \quad \text{so that}
\]

\[
w = x + \nu \in A.
\]

Then \( \delta I \leq I[w, J] A \geq 0 \).

So \( \delta I = \int_0^t L(x + \nu, x + \nu) ds \) has a min at \( \nu = 0 \)

\[\Rightarrow \quad \delta I(0) = 0 \quad \text{if } \nu \text{ exists}.
\]

\[
i(t) = \int_0^t L(x + \nu, x + \nu) ds
\]

\[
\Rightarrow \quad i'(0) = 0
\]

\[
i(t) = \int_0^t D_y L(x + \nu, x + \nu) \nu + DL(x + \nu, x + \nu) \nu ds
\]

set \( \nu = 0 \) ,

\[
\Rightarrow \quad i(0) = 0.
\]
\[ 0 = \Psi(0) = \int_0^t D\Psi(L(x, t)) + D\Phi(x) \Psi \, ds \]

Integrate by parts using

\[ \Psi(0) = \Psi(T) = 0 \]

\[ \int_0^t \left( -\frac{d}{ds} (D\Phi(L(x, t))) + D\Phi(L(x, t)) \right) \Psi \, ds \]

Since this is zero for all \( \Psi \in C^0_c((0, T); \mathbb{R}) \),

we obtain

\[ \int \left( -\frac{d}{ds} (D\Phi(L(x, t))) + D\Phi(L(x, t)) \right) \Psi \, ds = 0 \]

for \( \Psi \in C^0_c((0, T); \mathbb{R}) \)

**Example 1:**

\[ L(q, x) = \frac{1}{2} m(q(1)^2 - \Phi(x)) \]

\( m \) is kinetic energy

\( \Phi(x) \) is potential energy

\( F - L \) eqn is Newton's Law

\[ m \ddot{x} = -\Phi'(x) \]

**Example 2:**

\[ L(q, x) = \begin{cases} 0 & q_1 \leq 1 \\ \infty & q_1 > 1 \end{cases} \]

Cost functional is minimal time to move from \( y \) to \( x \)

\( v \) speed \( \leq 1 \). \( F - L \) eqn is harder to interpret.
A payoff problem

Define

\[ w(x, t) = \inf \left\{ \int_0^t L(w(s), w'(s)) \, ds + g(w(t)) \mid w(0) = y, w(t) = x \right\} \]

as the payoff/terminal cost

The value function

Optimal trajectories solve the Euler-Lagrange Eqn

\[ \frac{d}{ds} (\omega(t, x)) = 0 \quad 0 \leq s \leq t. \]

Dynamic programming principle

\[ u(x, t) = \inf \left\{ \int_t^\infty L(w(s), w'(s)) \, ds + u(x, s) \mid w(s) = y, w(t) = x \right\} \]

using the DPP infinitessimally

\[ u(x, t+\delta) = \inf \left\{ \int_t^{t+\delta} L(w(s), w'(s)) \, ds + u(x, s) \mid w(s) = y, w(t+\delta) = x \right\} \]

= \inf \left\{ \inf_v \left[ \inf_{t+\delta} \left\{ \int_t^{t+\delta} L(v(s), v'(s)) \, ds + u(x, s) \right\} \right] \right\} + O(\delta^2)

= \inf \left\{ \inf_v \left[ \inf_{t+\delta} \left\{ \int_t^{t+\delta} L(v(s), v'(s)) \, ds + u(x, s) \right\} \right] \right\} + O(\delta^2)

= \inf \left\{ \inf_v \left[ \int_t^{t+\delta} L(v(s), v'(s)) \, ds + u(x, s) \right] \right\} + O(\delta^2)
on

\[ \mathcal{Q} u = \inf \left\{ -u \cdot D u(x,t) + L(u) \right\} \]

so

\[ \mathcal{Q} u + \sup \left\{ v \cdot D u(x,t) - L(v) \right\} = 0 \]

\[ u \in \mathcal{H}(D u, x) \]

---

**Proof of DPP:**

\[ u_{x,t} = \inf \left\{ \int_0^t L(w, \dot{w}) \, dt + g(y) \middle| w(0) = y, \ w(t) = x \right\} \]

Let \( y \in \mathbb{R}^n \), \( x \in \mathbb{R}^n \), take \( u \) optimal for \( u(y,s) \), i.e.

\[ u(y,s) = \int_0^s L(w^*, w^*) \, ds + g(w^*(0)) \]

and let

\[ \overline{w^*} := \begin{cases} w^*(t) & s \leq t \leq t \\ w^*(s) & 0 \leq s \leq s \end{cases} \]

with \( \overline{w} \) any path \( \overline{w} \) s.t. \( w(s) = y \), \( w(t) = x \)

Then

\[ u(x,t) \leq \int_0^t L(\dot{\overline{w}}, \overline{w}) \, dt + g(\overline{w}(0)) \]

\[ = u(y,s) + \int_s^t L(\dot{w}^*, w(\cdot)) \, dt \]

Taking \( \inf \) over \( w \) yields

\[ u(x,t) \leq \inf \left\{ \int_0^t L(\dot{w}^*, w(\cdot)) \, dt + u(y,s) \middle| w(0) = y, \ w(t) = x \right\} \]
for the other direction let $u^*_t$ optimal

\[
\begin{align*}
\text{for } u_{x,t} &:= \int_0^t L(w_0, w_0) \, dt + g(w_t(0)) \\
&= \int_0^t L(w_t, w_t) \, dt + \int_0^S L(w_t, w_s) \, ds + g(w_t(0)) \\
&\geq \int_0^t L(w_t, w_t) \, dt + u(c_{x}(s), s) \\
\Rightarrow \quad u(x, t) &\geq \inf \left\{ \int_0^t L(w, w) \, dt + u(y, s) \mid \begin{array}{c}
w(0) = y, \ w(t) = x \end{array} \right\}
\end{align*}
\]
Suppose that $x$ is an action minimizer (or critical point) (call $p(s) = D_q L(x(s), x'(s))$ odd).

$p(-1)$ is called the generalized momentum.

$x$ - position, $x'$ - velocity.

Now we suppose that $\forall x, p \in \mathbb{R}^n$,

$$p = D_q L(v, x)$$

can be uniquely solved for $v$ as a smooth function of $p, x$

$$v = v(p, x).$$

Def. The Hamiltonian associated with the Lagrangian $L$ is

$$H(p, x) = p \cdot v(p, x) - L(v(p, x), x).$$
Legendre Transform

Now we put some additional assumptions

1. \( \forall q \in \mathbb{R}^n \), \( L(q) \) is convex

2. \( \lim_{|q| \to \infty} \frac{L(q)}{|q|} = +\infty \) (coercivity)

Def. The Legendre transform of \( L \) is

\[
L^*(p) = \sup_{q \in \mathbb{R}^n} \left\{ p \cdot q - L(q) \right\} \quad (p \in \mathbb{R}^n)
\]

Suppose the supremum is achieved by \( q_* \in \mathbb{R}^n \)

\[
L^*(p) = p \cdot q_* - L(q_*) \quad \text{and}
\]

\[
q_* \mapsto p \cdot q - L(q) \quad \text{has maximum at } q_*
\]

Then \( p = DL(q) \) is solvable for \( q \)

\[
\text{Uniquely} \quad L^*(p) = p \cdot q(p) - L(q(p)) \quad (p \cdot q(p) = L(q(p)))
\]
which is the Hamiltonian $H$ associated with $L$.

we thus call $H = L^*$. 

Now conversely given Hamiltonian $H$ can we find $L$?

Then (Convex duality) Assume $L$ convex coercive

and define $H = L^*$ then

1. $H$ is convex and coercive

2. $H^* = L$

Remark: We call $H$ and $L$ convex dual family

$p \mapsto p \cdot q - L(q)$ is linear

so $H(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(q) \}$ is convex.

$H(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(q) \}$

Take $q = \frac{R \ell_p}{\ell_p} \geq R \ell_p - L(\ell_p)$

$\geq R \ell_p - \max_{\ell(q) \in \ell_p} L(\ell(q))$ (coercive)

$\Rightarrow \lim_{\ell_p \to \infty} \frac{H(p)}{\ell_p} \geq R$ for every $R > 0$
\[(\text{duality}): \forall \ p, q \]

\[H(p) + L(q) \geq p \cdot q\]

so

\[L(q) \geq \sup_{r \in \mathbb{R}^n} \{ p \cdot q - H(p) \} = H^*(q)\]

and

\[H^*(q) = \sup_{p \in \mathbb{R}^n} \inf \{ p \cdot q - \sup_{r \in \mathbb{R}^n} \{ r \cdot p - L(r) \} \} = \sup_{p \in \mathbb{R}^n} \inf_{r \in \mathbb{R}^n} \{ p \cdot (q - r) + L(r) \}\]

Since \(L\) convex and it has a supporting hyperplane at \(q\), \(\exists r \in \mathbb{R}^n\) such that

\[L(r) \geq L(q) + s \cdot (r - q) \quad \forall r \in \mathbb{R}^n\]

taking \(p = s\) in the suprema

\[H^*(q) \geq \inf_{r \in \mathbb{R}^n} \{ s \cdot (q - r) + L(r) \}\]

\[\geq L(q) \quad \text{from choice of } s \in \mathbb{R}\]
The Hopf-Lax formula defines
\[ u(x,t) = \inf \left\{ \int_0^t L(\tilde{w}(s))\,ds + g(y) \mid \tilde{w}(0) = y, \tilde{w}(t) = x \right\} \]

will turn at \( t \) solve (in a sense)
\[
\begin{align*}
  u_t + H(Du) &= 0 & \text{in} & \ (\mathbb{R}^n \times (0,T)) \\
  u(x,0) &= g(x) 
\end{align*}
\]

where \( H = L^x \) or \( L^{ft} \) depending on our starting point.

This (Hopf-Lax formula)

but \( x \in (\mathbb{R}^n \setminus \Omega) \), \( t > 0 \) then \( u \) as \( \in C_1 \) is

\[
  u(x,t) = \min_{y \in \Omega} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}
\]

(i.e. optimal trajectories in \( C_1 \) are straight lines)
proof of Hopf-Lax formula:

one direction is easy

take \( W(x) = y + \frac{\varepsilon}{t}(x-y) \)

\[
W(x) \leq \int_0^t L(W(s))ds + g(y)
\]

\[
= \int_0^t L\left(\frac{x-y}{t}\right)ds + g(y)
\]

\[
= t L\left(\frac{x-y}{t}\right) + g(y)
\]

take \( \inf \) over \( x \rightarrow \infty \) to obtain

\[
\liminf_{x \rightarrow \infty} \leq \inf_{y \in \mathbb{R}} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\}
\]

if \( W \) is any \( C^1 \) \( \text{for} \ W(t) = x \) then

Calling \( y = W(0) \)

\[
L\left(\int_0^{x-t} L(W(s))ds\right) \leq \frac{1}{t} \int_0^t L(W(s))ds \quad \text{[from]}
\]

so

\[
\liminf_{x \rightarrow \infty} \geq \inf_{y \in \mathbb{R}} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\}
\]

\[
= \inf_{y \in \mathbb{R}} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\}
\]

([US]) super-linear growth of \( L \) to show \( \inf \) is achieved.
Lemma (Dynamic Programming Principle)

For each \( x \in \mathbb{R}^n \), \( 0 \leq s < t \) we have

\[
u(x_t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s) L \left( \frac{x-y}{t-s} \right) + \nu(y, s) \right\}
\]

Proof: 1. Fix \( y \in \mathbb{R}^n \), \( 0 < s < t \) and choose \( \tilde{z} \) so

\[
u(y,s) = (t-s) L \left( \frac{y-\tilde{z}}{t-s} \right) + g(\tilde{z})
\]

Since

\[
L \left( \frac{x-z}{t} \right) \leq (1 - \frac{s}{t}) L \left( \frac{x-y}{t-s} \right) + \frac{s}{t} L \left( \frac{y-z}{s} \right) \Rightarrow
\]

\[
u(x_t) \leq (t-s) L \left( \frac{x-y}{t-s} \right) + \frac{s}{t} L \left( \frac{y-z}{s} \right) + g(\tilde{z})
\]

for each \( y \in \mathbb{R}^n \).

Now choose \( w \) s.t.

\[
u(x_t) = (t-s) L \left( \frac{x-w}{t-s} \right) + g(w)
\]

for each \( y \in \mathbb{R}^n \).