A path in a metric space $X$ is a continuous function

$$\gamma : [0,1] \rightarrow X$$

such that

$$\gamma (0) = x, \quad \gamma (1) = y.$$ 

A space $X$ is called path connected if for every $x, y \in X$ there exists a path $\gamma$ in $X$ from $x$ to $y$.

**Theorem**: If $E \subset \mathbb{R}^k$ is open and $E$ is path connected, then $E$ is connected.

**Proof**: Suppose $E$ is not connected, then there exist open sets $U, V$ in $\mathbb{R}^k$ such that $E = U \cup V$ and $U \cap V = \emptyset$, then $E$ is not path connected.
Let $x \in U \cap E$ and $y \in U \cap E$

and $\gamma$ a path from $x$ to $y$

since the image of connected set under an is connected $\gamma([0,1])$

is connected, but $U \cup V$

separate $\gamma([0,1])$, as well since $x \notin U \cap \gamma([0,1])$ and $y \notin U \cap \gamma([0,1])$

$\implies$ so $E$ not path connected.

Now suppose $E$ is connected and

$\exists \ x, y \in E$ with no path in $E$ between them.

Call

\[ U = \{ z \in E : \exists \text{ path from } x \text{ to } z \text{ in } E \} \]

\[ V = \{ z \in E : \exists \text{ path from } y \text{ to } z \text{ in } E \} \]

\[ W = \{ z \in E : z \text{ does not have a path to } x \text{ or } y \text{ in } E \} \]
\[ \text{Huh, } U, V, W \text{ are open, all disjoint.} \]

Thus, \(W\) now and \(V\) would separate \(E\) (which is a contradiction).

1. \(U\) open,

suppose \(z \in E \Rightarrow \exists \text{ a path } \gamma \text{ from } x \text{ to } z, \]

let \(\delta > 0\) small enough that \(B(z, \delta) \subseteq E \subseteq B(0, \delta)\) open.

For \(z' \in B(z, \delta)\), define

\[ \gamma(t) = \begin{cases} R \left( \frac{t}{1 - \delta} \right) & \text{for } 0 \leq t \leq 1 - \delta \\ z + (t - (1 - \delta))(z' - z) & \text{for } 1 - \delta \leq t \leq 1 \end{cases} \]

Which is now a \(C^1\) path from \(x\) to \(z'\).

\(z' \in U \Rightarrow B(z', \delta) \subseteq U\)

\(\therefore U\) is open. \(Q.E.D.\)
Same argument $\Rightarrow$ $V$ is open.

Since $B = U \cup U \cup W$ and all disjoint and $B$ open,

for $W$, since $E$ is open, $B \ni x \cong B(\delta,\epsilon) \subseteq E$

therefore $B(\delta,\epsilon) \cap U \cap B(\delta,\epsilon) \cap W$

nonempty then we could make a path in $E$ from $x$ or $y$ resp.

to $z$ as before which

Contradiction $z \in W \subseteq B(\delta,\epsilon) \cap W$.

Similar argument for $U, V, W$ disjoint.

If $z \in U \cap V$ then $E \ni x, y \cong y \in$ path $x \to z = x, y \in$ path $y \to z$
\[ \gamma(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2((-2t+1) \frac{1}{2}) & \frac{1}{2} \leq t \leq 1 \end{cases} \]

is a path from \( x \) to \( y \) which we assumed did not exist.

So \( U \cup V \) separate \( E \)

which is a contradiction. \( \blacksquare \)

**Example:** Metric space which is connected but not path connected.

Let \( X = \) the closure in \( \mathbb{R}^2 \) of

\[ \{ \alpha \gamma(0, \frac{1}{2}) = x \in \mathbb{R}^2 \} \]

with the Euclidean metric.