Linear Space (Vector Space)

over field $\mathbb{R}$ (or $\mathbb{C}$ for more general)

is a span $V$ of addition and scalar mul $

$ for $u, v, w \in V$ and $a, b \in \mathbb{R}$ (or $\mathbb{C}$)

(i) $u + (v + w) = (u + v) + w$ (assoc.)
(ii) $u + v = v + u$ (commut)
(iii) $\exists e \in V$ s.t. $0 + u = u \neq V$
(iv) $\exists u \in V$ s.t. $u + (-u) = 0$
(v) $a(bv) = (ab)v$
(vi) $1v = v$
(vii) $a(u + v) = au + av$
(viii) $a(u + v) = au + av$

Examples
- $\mathbb{R}^k$ for $k \in \mathbb{N}_+$
- $\mathbb{R}^+$ = \{$(a_1, a_2, \ldots)$ : $a_i \in \mathbb{R}$\}
- $\mathbb{C} \mathbb{K}^1 = \{f : \mathbb{K} \to \mathbb{R} : f$ is cont} w/ $\mathbb{K}$ cont metric

space.
A normed linear space is a linear space $V$ with a norm $\| \cdot \| : V \to [0, \infty)$ satisfying:

1. $\|x\| > 0$ if $x \neq 0$
2. $\|ax\| = |a| \|x\|$ for $a \in \mathbb{R}$
3. $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)

Remark: Every normed linear space $V$ is a metric space with metric

$$d(x, y) = \|x - y\|$$

1. $\|x - y\| = 0$ if $x - y = 0$
2. $\|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\|$.
Examples:

- $\mathbb{R}^n$ w/ $\|x\|_2 = (\sum_{i=1}^{n} x_i^2)^{1/2}$
- $\mathbb{R}^n$ w/ $\|x\|_p = (\sum_{i=1}^{n} x_i^p)^{1/p}$ for any $1 \leq p < \infty$
  
  $\|x\|_\infty = \max \{ |x_1|, \ldots, |x_n| \}$
- $\mathbb{R}^n_+$ w/ $\|x\|_p = (\sum_{i=1}^{\infty} x_i^p)^{1/p}$ $1 \leq p < \infty$
  
  $\|x\|_\infty = \sup \{ x_i \}$

Called $L^p(\mathbb{R}^n_+)$ space.

- $C(K)$ w/ $\|f\| = \sup \{ |f(x)| \}$ on compact sets are bounded

Proof. (Hölder's Ineq.) If $p > 1$, $q > 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$ then for any sequence $x_1, y_1 \in \mathbb{R}$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} (\sum_{k=1}^{\infty} |y_k|^q)^{1/q}$$
proof: if \( \|x\|_p \) or \( \|y\|_p = \infty \) it is obvious.

If \( \|x\|_p \) or \( \|y\|_p = 0 \) it is also obvious.

Now by looking at

\[
x_k' = \frac{x_k}{\|x\|_p} \quad y_k' = \frac{y_k}{\|y\|_p}
\]

Can assume that \( \|x\|_p = 1 \), \( \|y\|_p = 1 \)

then use Young's inequality

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}
\]

for \( \frac{1}{p} + \frac{1}{q} = 1 \)

\[\text{proof: using that log is (concave)}\]

\[
\log \left( \frac{1}{p} a^p + \frac{1}{q} b^q \right) \geq \frac{1}{p} \log a^p + \frac{1}{q} \log b^q
\]

(since \( \frac{1}{p} + \frac{1}{q} = 1 \) )

\[= \log(ab)\]

\[
\sum_{k=1}^{N} |x_k y_k| \leq \sum_{k=1}^{N} \frac{|x_k|^p}{p} + \frac{|y_k|^q}{q} = \frac{1}{p} \sum_{k=1}^{N} |x_k|^p + \frac{1}{q} \sum_{k=1}^{N} |y_k|^q
\]

\[\leq \frac{1}{p} + \frac{1}{q} = 1\]

so

\[
\sum_{k=1}^{N} |x_k y_k| \leq 1
\]
The (Minkowski's Ineq) for \( x, y \in \mathbb{R}^n \) or \( C(\mathbb{R}, ..., \mathbb{R}) \)

\[ \|x + y\|_p \leq \|x\|_p + \|y\|_p \]

Proof:

\[
\sum_{k=1}^{N} (x_k + y_k)^p = \sum_{k=1}^{N} |x_k + y_k|^p \leq \sum_{k=1}^{N} (|x_k| + |y_k|) |x_k + y_k|^{p-1} \\
= \sum_{k=1}^{N} |x_k| |x_k + y_k|^{p-1} + \sum_{k=1}^{N} |y_k| |x_k + y_k|^{p-1}
\]

(Averd Ineq) \( \left( \frac{1}{q} \right) \leq \left( \sum_{k=1}^{N} |x_k|^p \right)^{1/p} \left( \sum_{k=1}^{N} |y_k|^p \right)^{1/q} \left( \sum_{k=1}^{N} (|x_k + y_k|^{p-1}) \right)^{\frac{1}{p-1}} \)

\[
\left( \frac{1}{q} \right) \leq \left( \sum_{k=1}^{N} |x_k|^p \right) \left( \sum_{k=1}^{N} |y_k|^p \right) \left( \sum_{k=1}^{N} (|x_k + y_k|^{p-1}) \right)^{\frac{1}{p-1}}
\]

So if dividing throughout by \( \left( \sum_{k=1}^{N} (|x_k + y_k|^{p-1}) \right)^{\frac{1}{p-1}} \)

\[
\left( \sum_{k=1}^{N} |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{N} |x_k|^p \right)^{1/p} + \left( \sum_{k=1}^{N} |y_k|^p \right)^{1/p}
\]

Then let \( N \to \infty \) on left and right...
So $L^p$ spaces are indeed normed linear spaces. In particular, they are metric spaces as well.

Let's show that

$\overline{B(0,1)} = \{ x \in L^p(\mathbb{N}) : \| x \|_p \leq 1 \}$ in $L^p(\mathbb{N})$

is not compact.

Proof: I'll show that there is an infinite sequence with no convergent subsequence.

Let $x_n = x_n^k$; $x_n^k = \{ 0 \text{ for } n \notin k \}$

In

$x_1 = (1,0,\ldots,)$

$x_2 = (0,1,0,\ldots)$

$x_3 = (0,0,1,0,\ldots)$
All $x^n$ have $\|x^n\|_p = 1$

but also $\|x^n - x^m\| = 1$ for all $n \neq m$

$\Rightarrow x^n$ not Cauchy

$\Rightarrow x^n$ does not converge.

Closed and bounded sets are not always compact in general.

A Banach space is a complete normed linear space.

Then $l^p(N)$ is complete for a Banach space for every $1 \leq p < \infty$.

proof: let $x^n$ Cauchy in $l^p(N)$

since $\|x^n(i) - x^m(i)\| \leq \|x^n - x^m\|_p$

Can show $x^n(i)$ Cauchy in $\mathbb{R}$ for all $i \in \mathbb{N}$. 
Col1 \( X(i_n) = \lim_{n \to \infty} X^n(i) \) which exists since \( M \) is complete

Claim: that \( X^n \to X \) in \( L^p \)

Let \( \varepsilon > 0 \) for \( s.t. \)

\[
\left( \sum_{k=1}^{\infty} \left| X_k(i) \right|^p \right)^{1/p} \leq \varepsilon
\]

First, let's show that \( X \) is actually in \( L^p \)

\[
\left( \sum_{k=1}^{\infty} \left| X(i) \right|^p \right)^{1/p} < \infty
\]

\( \forall \varepsilon > 0 \) \( \exists N \) \( \forall i, n, m \in \mathbb{N} \)

\[ \| X^n - X^m \|_p = \left( \sum_{j=1}^{\infty} |X^n_j - X^m_j|^p \right)^{1/p} < \varepsilon \]

Since partial sums are monotonic \( \forall M \in \mathbb{N} \)

\[ \sum_{j=1}^{M} |X^n_j - X^m_j|^p < \varepsilon \quad \text{for} \quad n, m \in \mathbb{N} \]

Then send \( m \to \infty \) since LHS is finite sum of Cts fans \((lX_j^n - 0)^p\)

is Cts \( \forall j \) was
\[ \sum_{j=1}^{\infty} |x^n_j - x_j|_p < \varepsilon \quad \text{for every } m, n \geq N \]

Then send \( M \to \infty \), \( \varepsilon \to 0 \) for partial sum so the limit

\( (\text{which is their sup since terms are positive}) \text{ sequence of partial sum) is monotonically increa} \)

\[ \sum_{j=1}^{\infty} |x^n_j - x_j|_p < \varepsilon \quad \text{for all } n \geq N \]

\[ \|x^n - x\|_p < \varepsilon^{1/p} \quad \text{for all } n \geq N. \]

This means that \( x \in \ell^p(\mathbb{N}) \) since

\[ \|x\|_p \leq \|x - x^n\|_p + \|x^n\|_p \leq 1 + \|x^n\|_p < \infty. \]

And

\[ x^n \to x \quad \text{in } \ell^p \text{ metric.} \]

Thus every Cauchy sequence in \( \ell^p(\mathbb{N}) \) converges

\[ \Rightarrow \ell^p(\mathbb{N}) \text{ is complete. } \]
The $C(K)$ with sup norm is complete (i.e., it is a Banach space).

**Proof:** Let $f_n$ be a Cauchy sequence in $C(K)$.

Since $f_n$ is Cauchy, for $k \geq N$ we have $n, m \geq N$.

\[ \|f_n - f_m\|_{C(K)} = \sup_{x \in K} |f_n(x) - f_m(x)| < \varepsilon \]

So in particular, since for any fixed $t \in K$

\[ |f_n(t) - f_m(t)| \leq \|f_n - f_m\|_{C(K)} < \varepsilon \]

for $n, m \geq N$ as above

$\Rightarrow$ fully Cauchy in $\mathbb{R}$

So it converges. Call the limit $f(t)$.

Let $\varepsilon > 0$, $\exists N$ s.t. for all $n, m \geq N$

\[ |f_n(t) - f_m(t)| < \varepsilon \]

for all $t \in K$

Take the limit as $n \to \infty$

\[ |f(t) - f(t)| < \varepsilon \]

for all $t \in K$, $n \in \mathbb{N}$.
So in other words

$$\|f_n - f\|_{C(K)} = \sup_{t \in K} |f_n(t) - f(t)| < \epsilon$$ for \(n \geq N\).

\[\Rightarrow \quad f_n \to f \quad \text{in \(\|\cdot\|_{C(K)}\) norm.}\]

As we will show later this actually means that \(f\) is a continuous function as well.

\[\square\]

**Theorem:** If \(\|f_n - f\|_{C(K)} \to 0\) as \(n \to \infty\) for \(f_n \in C(K)\), \(f: K \to \mathbb{R}\) then \(f\) is continuous on \(K\).

**Proof:** Let \(x \in K\), \(\epsilon > 0\), \(N \in \mathbb{N}\) s.t.

\[\forall n \geq N \Rightarrow \quad \|f_n - f\|_{C(K)} < \frac{\epsilon}{3}\]

\[\Rightarrow \quad \exists \delta > 0 \text{ s.t. } d(x, y) < \delta \Rightarrow \quad \|f_n(x) - f_n(y)\| < \frac{\epsilon}{3}\]

\[\Rightarrow \quad 1 \quad \|f_n(x) - f(y)\| < \frac{\epsilon}{3}\]
\[ |f_{m+1}(x) - f_{m}(x) - f_{m}(y) + f_{m}(y)| \leq \frac{2}{3} + \frac{3}{3} + \frac{2}{3} = 3 \]

Space which is not complete

We could put a different norm on \( C[0, 1] \)

Called the \( L^1 \) norm

\[ \|f\|_{L^1} := \int_0^1 |f(x)| \, dx \]

This normed space \( (C[0, 1], \| \cdot \|_1) \)

is not a Banach space because it is not complete.
Proof: Take \( f_n(t) = \begin{cases} t^n & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \end{cases} \)

As \( n \to \infty \), \( t^n \to 0 \) for every \( t \in (0,1) \), so if it were to converge,

\[ f_n \to h, \quad h(t) = \begin{cases} 0 & 0 < t < 1 \\ 1 & 1 < t < 2 \end{cases} \]

but that function is not \( C^1 \).

Let's show that \( f_n \) is in fact Cauchy in the \( L^1 \) metric.

\[ \| f_n - f_m \|_{L^1([0,2])} = \int_0^1 t^n - t^m \, dt \]

Take \( n \neq m \),

\[ \int_0^1 (t^n - t^m) \, dt = \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1} \to 0 \quad \text{as} \quad n \to \infty. \]

Now let's show that \( f_n \) cannot possibly converge to any \( C^1 \) function in \( L^1 \) metric...

Take being how careful since we don't know what beautiful integration yet anyway.
Another Example

Let \( l_0(N) = \{ \{a_1, a_2, \ldots \} : \text{only finitely many } a_j \neq 0 \} \)

Can make \( l_0(N) \) into a normed linear span with any of the \( l_p \) norms

\(( l_0(N), \| \cdot \|_p \) \)

This space is not complete

E.g.

Given \( x \in l_0(N) \), take

\[ x^n = (x_1, x_2, \ldots, x_n, 0, \ldots) \in l_0(N) \]

For \( n < m \),

\[ \| x^n - x^m \|_p = \left( \sum_{k=m}^{\infty} \left| x_k \right|^p \right)^{1/p} \]

Let \( N \) so large that \( \forall n \geq N \)

\[ \sum_{k=n}^{\infty} \left| x_k \right|^p < \epsilon^p \] \( \text{(allowed sum sum converges)} \)

Then for \( n \geq N \)

\[ \| x^n - x^m \|_p < \epsilon \]
On the other hand $\ell^0(N)$ is dense in every $\ell^p(N)$.

Then $\ell^0(N)$ is a dense subset of $\ell^p(N)$.

**Proof:** Let $x \in \ell^p(N)$ and $\varepsilon > 0$

Find $N \in \mathbb{N}$ s.t. \( \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \varepsilon \)

Take $y = (x_1, \ldots, x_N, 0, \ldots) \in \ell^0(N)$

\[ ||y - x||_{\ell^p} \leq \left( \sum_{n=1}^{N} |x_n|^p \right)^{1/p} < \varepsilon \]

Notice that $\ell^0(N)$ is not countable.

Since $\ell^0(N)$ is not countable

Remember $\ell^0(N)$ has no countable dense

subspace --- it is a separable

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Non-separable Metric Space

\( L^\infty (\mathbb{N}_+) = \{ x \in \mathbb{R}^{\mathbb{N}_+} : \sup_{i} |x_i| < +\infty \} \)

With norm \( \| x \|_\infty = \sup_{i \in \mathbb{N}_+} |x_i| \)

\( \| x \|_\infty \cdot \| y \|_\infty = \sup_{i \in \mathbb{N}_+} \{ |x_i + y_i| : i \in \mathbb{N}_+ \} \)

\( \leq \sup_{i \in \mathbb{N}_+} \{ |x_i| + |y_i| : i \in \mathbb{N}_+ \} \)

\( \leq \sup_{i} |x_i| + \sup_{i} |y_i| \)

\( = \| x \|_\infty + \| y \|_\infty \)

\( L^\infty (\mathbb{N}_+) \) is a Banach space but it is not separable.

Thus \( L^\infty (\mathbb{N}_+) \) has no countable dense subset.

Proof: This will be like the Cantor diagonal argument.

Suppose \( x_1, x_2, x_3, \ldots \) is a countable dense set of \( L^\infty (\mathbb{N}_+) \).
We will construct a point \( y \) s.t. 
\[
\| x^a - y \|_\infty = 1 \quad \text{for all} \quad a = 1, 2, \ldots
\]
so that \( B(y, 1) \cap \{ x^a : a \in \mathbb{N}_+ \} = \emptyset \)

which contradicts that \( x^a, x^b \) are dense.

For \( c \in \mathbb{N}_+ \), define \( y_i = x_i^c + 1 \) so that
\[
l = \| y_i - x_i^c \|_\infty \leq \sup_j \| y_j - x_j^c \| = \| x^b - y \|_\infty
\]

5. \( \| x^a - y \|_\infty = 1 \quad \forall \quad a \in \mathbb{N}_+ \). \( \square \)