Problem 1. Chapter 2 (p. 44): 17, 18, 21
Problem 2. Chapter 3 (p. 79): 16, 19
Problem 3. Chapter 4 (p. 98): 6, 9, 11, 25
Problem 4. Suppose that \( \{x_n\}_{n=1}^{\infty} \) is a sequence in a metric space \( X \) with the following property: there is \( x \in X \) such that every subsequence of the \( x_n \) has a subsequence converging to \( x \). Show that \( x_n \) converges to \( x \).

Problem 5. Given a function \( f : \mathbb{R}^k \to \mathbb{R} \) define the support of \( f \)

\[
\text{supp}(f) = \{ x \in \mathbb{R}^k : |f(x)| > 0 \}.
\]

A function \( f \) is said to have compact support if \( \text{supp}(f) \) is a compact set of \( \mathbb{R}^k \). We define the vector space of continuous functions with compact support on \( \mathbb{R}^k \) (why is it a vector space? check for yourself you don’t need to write it):

\[
C_c(\mathbb{R}^k) = \{ f : \mathbb{R}^k \to \mathbb{R} : f \text{ is continuous and has compact support} \}.
\]

Show that \( C_c(\mathbb{R}^k) \) with the norm \( \|f\| := \sup_{x \in \mathbb{R}^k} |f(x)| \) is a normed space (i.e. check that \( \|f\| < +\infty \) for all \( f \in C_c(\mathbb{R}^k) \)). Show by an example that this normed space is not complete.

The following final part is just for your own interest you don’t have to do it: show that the completion of \( C_c(\mathbb{R}^k) \) under the supremum norm \( \|\cdot\| \) is the space \( C_0(\mathbb{R}^k) \) of continuous function which vanish at \( \infty \),

\[
C_0(\mathbb{R}^k) := \{ f : \mathbb{R}^k \to \mathbb{R} : f \text{ is continuous and } \lim_{|x| \to \infty} f(x) = 0 \}.
\]

Problem 6. Two metrics \( d_1 \) and \( d_2 \) on a space \( X \) are said to be equivalent if for any sequence \( \{x^j\}_{j=1}^{\infty} \) in \( X \),

\( x^j \) converges to \( x \) in \( d_1 \) metric if and only if \( x^j \) converges to \( x \) in \( d_2 \) metric.

(i) Show that all the \( \ell^p \) distances for \( 1 \leq p \leq \infty \) on \( \mathbb{R}^k \) are equivalent. Recall that

\[
d_{\ell^p}(x, y) = \left( \sum_{n=1}^{k} |x_n - y_n|^p \right)^{1/p} \quad \text{and} \quad d_{\ell^\infty}(x, y) = \sup_{n=1,\ldots,k} |x_n - y_n|.
\]

Hint: Show that convergence in any \( d_{\ell^p} \) is equivalent to convergence of all the entries in absolute value.

(ii) Now consider the sequence space,

\[
\ell^1(\mathbb{N}_+) = \{ x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}_+} : \|x\|_1 = \sum_{n=1}^{\infty} |x_n| < +\infty \}.
\]

Show that convergence with respect to \( \ell^1 \)-metric \( d_{\ell^1}(x, y) = \|x - y\|_1 \) implies convergence with respect to the \( \ell^\infty \) metric

\[
d_{\ell^\infty}(x, y) := \sup_{n \in \mathbb{N}_+} |x_n - y_n|.
\]

Give an example of a sequence in the space \( \ell^1(\mathbb{N}_+) \) which converges in \( \ell^\infty \) metric but not in \( \ell^1 \) metric.

Problem 7. [Tao, Analysis II, 12.5.10] A metric space \( (X, d) \) is called totally bounded if for every \( \varepsilon > 0 \) there exists a positive integer \( n \) and points \( x_1, \ldots, x_n \in X \) so that \( B(x_1, \varepsilon), \ldots, B(x_n, \varepsilon) \) cover \( X \).

(i) Show that a totally bounded space is bounded.
(ii) Show that if $(X, d)$ is sequentially compact (every sequence has a convergent subsequence) then it is complete and totally bounded. Hint: if $X$ is not totally bounded then there is some $\varepsilon > 0$ so that there is no finite covering of $X$ by balls of radius $\varepsilon$. Show that in this situation, given any finite collection of $\varepsilon/2$ radius balls there is another $\varepsilon/2$ radius ball which is disjoint from all of them. Use this inductively to construct an infinite sequence $B(x_j, \varepsilon/2)$ of disjoint balls. From here you can contradict sequential compactness.