Recall from last time:

We showed that for $(X,d)$ complete metric space

$$C(X) = \{ f : X \to \mathbb{R} : f \text{ is } \text{cts} \}$$

with

$$||f||_{\text{sup}} = \sup_{x \in X} |f(x)|$$

is a Banach space, but we need the following theorem:

**Theorem:** Suppose $f_n : X \to \mathbb{R}$ are cts and $f : X \to \mathbb{R}$ some function. If $||f_n - f||_{\text{sup}} \to 0$ as $n \to \infty$ then $f$ is cts as well.

**Proof:** Let $x \in X$, $\epsilon > 0$. Let $N$ s.th. $n \geq N$ implies

$$||f_n - f||_{\text{sup}} < \frac{\epsilon}{3}$$

then $f_n \text{ cts } \Rightarrow \exists \delta > 0 \text{ s.t. } d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\epsilon}{3}$

$$\Rightarrow \quad |f_n(x) - f_n(y)| < \frac{\epsilon}{3}$$

$$\Rightarrow \quad |f_n(x) - f_n(y)| < \frac{\epsilon}{3}$$
Then $d(y, x) < s \Rightarrow$

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$
Compactness Revisited

A set \((X,d)\) is called totally bounded if for all \(\varepsilon > 0\), there exists \(x_1, \ldots, x_n \in X\) such that

\[
\exists \varepsilon \quad X = \bigcup_{i=1}^{n} B(x_i, \varepsilon).
\]

Then the metric space \((X,d)\) is compact if

(i) \(X\) compact

(ii) \(X\) sequentially compact

(iii) \(X\) complete and totally bounded.

I'm having you prove in the homework that

sequentially compact \(\Rightarrow\) complete and totally bounded.

We already showed

compact \(\Rightarrow\) sequentially compact

Let's show that complete \& totally bounded \(\Rightarrow\) compact.
First let's note that any subset of a totally bounded metric space is totally bounded.

Let $E \subseteq X$.

$X$ is totally bounded if $E \subseteq X = \bigcup_{i=1}^{n} B(x_i, \varepsilon)$.

Then $E = \bigcup_{i=1}^{n} B(x_i, \varepsilon)$.

Suppose $X$ is not compact so there is an open cover $\{G_d\}$ with no finite subcover.

Since $X$ is totally bounded we can cover $X$ by balls $B(x_i, \frac{1}{2})$ such that $X$ is covered with a diameter (at most) 1.
Since there are only finitely many such balls $\{G_\alpha\}$, we do not have a finite subcover for at least one of them. Call that set $E_1$.

Then apply the argument again inductively. Given $E_k$ define a cover $E_k$ by finitely many balls $B(x, \frac{1}{2^n}) \forall x \in E_k$. $E_k$ must not have a finite subcover for one of these, call it $E_{k+1}$.

In this way we get a nested sequence of sets $E_k$ of diameter $\leq \frac{1}{2^n}$ such that $\bigcap_{i=1}^{\infty} E_i$ is the desired set.
In particular each $E_k \neq \emptyset$.

Let $x_k \in E_k$, we claim $x_k$ is Cauchy. Let $\varepsilon > 0$ and $N > \frac{1}{\varepsilon}$.

For $n, m > N$, \[ x_n \in E_n \subseteq E_N, \quad x_m \in E_m \subseteq E_N \]

\[ d(x_n, x_m) \leq \text{diam } E_N \leq \frac{1}{N} \leq \varepsilon. \]

So, \[ \{x_n\}_{n=1}^{\infty} \] is Cauchy in $X$ and hence it converges to some $x_\infty$ (since $X$ complete).

Since \( \{G_n\} \) open cover of $X$, $x_\infty \in G_0$ for some $n_o$ and $\exists r > 0$ such that \[ B(x_\infty, r) \subseteq G_0. \]
Let \( N \geq \frac{1}{r} \) so then for all \( x \in E_N \)

(since \( x_0 \in \overline{E_n} \) for all \( n \) and \( \text{diam} \overline{E_n} = \text{diam} E_n \))

\[ d(x_0, x) \leq \text{diam} \overline{E_n} = \text{diam} E_n \leq \frac{1}{N} < r \]

so \( E_N \subseteq B(x_0, r) \leq G_{\alpha_0} \)

which is a contradiction of the choice of \( E_N \). \( \square \)
Recall that a metric space $(X,d)$ is called sequentially compact if every sequence in $X$ has a convergent subsequence.

Let's suppose some of our main theorems using the sequential compactness def'n (which we know is equivalent to open cover compactness) just to see how the def'n is used in practice.

Then let $E$ closed and $K$ compact metric space $(X,d)$.

Then $$\inf_{x \in E, y \in K} d(x,y) > 0.$$  \[\text{proof: Suppose otherwise then } A, \eta > 0 \text{ s.t.} x \in E, y \in K \text{ s.t.} d(x,y) < \frac{1}{n}, \frac{1}{n} \text{ not a LB for } \{d(x,y): y \in K\}.\]

$y \in K$ so has a convergent subseq $y_n \to y \in K$. 
Then we claim $y$ is a limit pt of $E$ \\

$$
\Rightarrow y \in \bigcap N \quad \text{which is a contradiction}
$$

Let $r > 0$, $\exists N \ni n \quad \text{for } k \geq N$

$$
N \ni r \quad \Rightarrow d(x_k, y_n) \leq \frac{r}{2} \quad \text{and} \quad d(y_k, y) \leq \frac{r}{2}
$$

so $d(x_n, y) \leq d(x_n, y_k) + d(y_k, y)$ \\

$$
\leq r
$$

Thus suppose $f$ is a cb map of a cbet metric space $X$ into a metric space $Y$. Then $f(x)$ is cbet.

**Proof:** Let $y_n$ be a sequence in $f(X)$ \\
i.e. $y_n = f(x_n)$ for some $x_n \in X$

$$
X \quad \text{each } \Rightarrow x \quad x_n \rightarrow x \quad x \in X
$$

$f$ is cbet $\Rightarrow f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$ \\
i.e. $y_n \rightarrow f(x) \in f(X)$ \(\Box\)
Then let \( f \) be a uniformly continuous map of a metric space \((X, d_X)\) into a metric space \((Y, d_Y)\).

Then \( f \) is uniformly cts.

**Proof:** Suppose otherwise. There \( \varepsilon > 0 \) s.t. \( A \in \mathbb{N} \)

where

There are \( p_n, q_n \in X \) w/ \( d_X(p_n, q_n) < \frac{1}{n} \) but \( d_Y(f(p_n), f(q_n)) > \varepsilon \).

Since \( X \) cpts, \( p_n \) has a subsequence of \( p_n \)

\( p_{n_k} \to p \)

First let's see that \( q_{n_k} \to p \) as well.

Let \( \varepsilon > 0 \) and \( k > \frac{2}{\varepsilon} \) (so that \( \frac{1}{n_k} > \frac{1}{2} \) as well)

Then \( d_Y(q_{n_k}, p) \leq d_Y(q_{n_k}, p_{n_k}) + d_Y(p_{n_k}, p) \)

\[ \leq \frac{1}{n_k} + \frac{\varepsilon}{2} \leq \varepsilon. \]
\[ f(p) = \lim_{n \to \infty} f(p_n) = f(p) = \lim_{n \to \infty} f(p_n) \]

\[ D = \lim_{n \to \infty} d(f(p_n), f(p_{n+1})) > \varepsilon \]

which is a contradiction \( \square \)