Compact Sets

This is an extremely important notion in analysis. The first definition we give may be a bit unintuitive.

We say that \( \{G_x\} \) is an open cover of a set \( E \subseteq X \), \((X,d_X)\) metric space, if each \( G_x \) open and \( E \subseteq \bigcup_{x \in E} G_x \).

We say that \( K \subseteq \) metric space \( X \) is compact if every open cover contains a finite subcover.

i.e., if \( \{E_x\} \) is an open cover of \( K \), then \( \exists \ \alpha_1, \ldots, \alpha_n \) s.t.

\[
K \subseteq \bigcup_{i=1}^{n} \bigcup_{x \in \alpha_i} G_x
\]
Compactness, unlike open/closed, behaves well w.r.t. relative-ness

**Theorem 2.30:** \( K \subseteq Y \subseteq X \). Then \( K \) compact rel. to \( Y \) \( \iff \) \( K \) compact rel. to \( X \).

This means that talking about a compact metric space makes sense.

**Proof:** I'll skip the proof read Thm's 2.30 and 2.33 in the book.

**Theorem:** Compact sets are closed.

**Proof:** Let \( K \) compact \( \subseteq X \) metric space.

We will show \( K^c \) open, if \( K = \emptyset \Rightarrow \emptyset \) open.

Let \( p \in K^c \). For each \( q \in K \) \( \exists \varepsilon > 0 \) (draw)

\[ B(p, \varepsilon) \cap K = \emptyset \]
\( V_q = B(c_p, \frac{d(c_p, q)}{2}) \)

\( W_q = B(q, \frac{d(c_p, q)}{2}) \)

which are disjoint

\[ \{W_q\}_{q \in K} \text{ covers } K \text{ (since } q \in W_q) \]

so \( \exists q_1, \ldots, q_n \text{ s.t. } \bigcup_{i=1}^n W_{q_i} \text{ covers } K \).

Let \( V = V_{q_1} \cap \ldots \cap V_{q_n} \) which is open

\( p \in V \) and

\[ V \subset \bigcap_{i=1}^n W_{q_i} \subset K^c \]

so \( p \) is an interior pt of \( K^c \).

\( \Box \)
Then: Closed subsets of compact sets are compact.

Proof: Let $E$ be closed and compact.

Let $\{G_x\}$ be an open cover of $E$.

$\bigcup_{x} G_x \subseteq E$ is an open cover of $E$.

$\Rightarrow E \subseteq G_1 \cup \cdots \cup G_n$, an open cover of $E$.

($E^c$ does not need to be in these since $E^c \cap E = \emptyset$)

Theorem: If $\{K_x\}$ is a collection of compact sets in metric space $(X,d)$ such that for every finite subcollection

$\bigcap_{x \in A} K_x \neq \emptyset$, then

$\bigcap_{x \in A} K_x \neq \emptyset$. 
proof: for each \( k \in \mathbb{N} \) put \( G_k = K_k \).

Suppose that \( \bigcap_{k \in \mathbb{N}} K_k \) is empty.

Then for each \( x \) left

\[
K_x \subset \bigcup_{k \in \mathbb{N}} G_k
\]

If a finite subcover \( G_{j_1}, \ldots, G_{j_n} \)

of \( K_x \)

but then \( K_{j_1} \cap \cdots \cap K_{j_n} \cap K_x = \emptyset \)

contradicting the finite intersection property.

Then (compact \( \Rightarrow \) sequentially compact)

If \( E \) is an infinite subset of a cnt space

then \( E \) has a limit point in \( X \).
Proof: Suppose otherwise. Pick for each $p \in K$ a neighborhood $B(p, r_p)$ such that $B(p, r_p) \cap E$ has at most one element. Then $K$ has a finite subcover and it has a finite subcover on $E \cap \bigcup_{p \in K} B(p, r_p)$. This contradicts $E$ being infinite. $\Box$
Now we begin on a proof of the following important theorem which shows that several definitions of compactness are equivalent.

**Theorem:** Let $E \subseteq \mathbb{R}^n$ a Euclidean metric space.

(i) $E$ closed and bounded
(ii) $E$ compact
(iii) Every infinite subset of $E$ has a limit point in $E$.

**Remark:** (i) and (iii) are equivalent in any metric space, but (i) in general is not.

We will just prove the theorem in $\mathbb{R}^n$.

Read the book for proof in $\mathbb{R}^n$. 
First let's show that closed intervals $[a, b] \subseteq \mathbb{R}$ are compact.

**Theorem (FIP for intervals):** If $I_n$ is a nested (Downward) sequence of closed intervals of $\mathbb{R}$ then

$$\bigcap_{n=1}^{\infty} I_n$$

is not empty.

**Proof:** $I_n = [a_n, b_n]$

Let $\mathcal{E} = \{a_n : n \in \mathbb{N}, n \rightarrow \infty\}$

$a_n \leq b_n$ for all $n$ so $\exists$ an odd above.

Call $x = \sup \mathcal{E}$.

Then $a_n \leq x$ for all $n$.

On the other hand

$$a_n \leq a_n + \epsilon \leq b_n \leq b_m$$

so $b_m$ are all UB for $\mathcal{E}$.
So \( x \leq b_m \) for all \( m \)

\[ \Rightarrow x \in I_n \quad \forall n \in \mathbb{N} \]

Remark: Closed bounded intervals \([a,b] \) are compact.

Proof: Suppose there is an open cover 
\[ \{ G_x \} \] and \( \{ \} \) no finite subcover.

Divide \( I \) into half

\[ [a, b] = \left[ a, \frac{a+b}{2} \right] \cup \left[ \frac{a+b}{2}, b \right] \]

\( \{ G_x \} \) covers left and right halves

and must fail to have a finite subcover for one of the two

Call that interval \( I_1 \).

Apply the same reasoning inductively to get

\[ I_j \quad j \in \mathbb{N} \text{ nested intervals} \]

\[ w \quad (b_j - a_j) = 2^{-j} (b-a) \]
by the NIP of intervals,

so \( \bigcap I_j \) is nonempty, containing some \( x_p \).

Since \( G_1 \) is open, \( \exists r > 0 \) such that

\[ B(x_p, r) \subseteq G_1. \]

Let \( j \) sufficiently large (by Archimedean property)

\[ 2^{-j} (b-a) < \frac{r}{2}. \]

Then \( x_p \in I_j \),

so \( b - x_p \leq 2^{-j} (b-a) < r \)

\( x_p - a \leq 2^{-j} (b-a) < r \)

so \( I_j \subseteq B(x_p, r) \subseteq G_1 \)

this is a finite subset of \( I_j \)

\( \Rightarrow \)
Theorem (Heine-Borel)

Let $E \subseteq \mathbb{R}$ (or $\mathbb{R}^n$) be a \textit{tube}.

\begin{enumerate}[(a)]
  \item $E$ is closed
  \item $E$ is compact
  \item Every infinite subset of $E$ has a limit point in $E$
\end{enumerate}

\textbf{Proof:}
(b) \implies (c) true in general metric space, but harder

If (a) holds then hold \textit{c}.

\textbf{Thus:}

(a) $E \subseteq [a,b]$ for some $a,b$

so closed subset of compact set is compact

$\implies$ (b)

\textbf{Hence (b) \implies (a):}

closed we know, also $E$ hold since for any fixed $x \in X$

\[ \{B(x, R) \}_{R>0} \] is an open cover of $E$.

Take finite subcover $\{B(x_i, R_i) \}_{i \geq 1}$ and let $R_x = \max_{i \geq 1} R_i$

then $E \subseteq B(x_i, R_x)$.

So (b) \implies (a)

Now let show (c) \implies (b)

\[ \text{(we already know (b) \implies (c))} \]
Finally we show (e) $\Rightarrow$ (a)

Suppose $E$ is not closed

$\Rightarrow \exists x_n \in E \forall n \in \mathbb{N}$

$\{x_1, x_2, \ldots\}$ is infinite

and does no limit points

so $E$ is total

Suppose $E$ is not closed $\Rightarrow \exists x \in \mathbb{R} \text{ limit pt of } E \text{ but not in } E$.

Since $x_1$ is a limit pt of $E$

A neib $E \ni x_n \in E$ w

$|x_n - x_1| < \frac{1}{n}$ (and $x_n \neq x_1$)

let $S = \{x_1, x_2, \ldots\}$

then $S$ is infinite

$S$ has $x_1$ as a limit point and no other $y \notin E$ since

$|x_n - y| = |x_n - x_1 + x_1 - y| = |x_n - x_1| + |y - x_1| - \frac{1}{n}$
which is \( \geq \frac{1}{2} |x_n - y| \)

for all but finitely many \( n \)

\( \Rightarrow \ y \) not a limit point of \( E \)

if \( y \neq x_k \)

Thus since \( S \) has a limit point in \( E \)

that must be \( x_k \) so \( x_k \in E \)

---

**Perfect Sets**

**Theorem** Let \( P \) be a perfect set in \( \mathbb{R}^n \). Then \( P \) is uncountable.

**Proof**: Since \( P \) has limit pts \( P \) must be infinite. Suppose \( P \) countable.

Enumerate \( P \) by \( x_1, x_2, \ldots \)

Consider \( \mathcal{V}_1 = B(x_1, r) \) for any some \( r > 0 \)

\[ \mathcal{V}_1 = \{ x : |x - x_1| \leq r \} \]
suppose \( V_n \) constructed and
\[ V_n \cap P \text{ non-empty} \]
since every pt of \( P \) is a limit pt of \( P \)
we can choose a nbhd \( V_{n+1} \) s.t.
\[ \overline{V_{n+1}} \subset V_n, \quad x \notin \overline{V_{n+1}} \text{ and } V_{n+1} \cap P \neq \emptyset \]
construct \( V_n \) inductively in this way
Call \( K_n = \overline{V_n} \cap P \) closed, bold and hence Cpt sets of \( \mathbb{R}^n \)
\[ x \notin \bigcap_{n=1}^{\infty} K_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} K_n \neq \emptyset \]
Since \( K_n \subset P \Rightarrow \bigcap_{n=1}^{\infty} K_n = \emptyset \)
but by since each \( K_n \) nonempty
\[ \bigcap_{n=1}^{\infty} K_n \text{ nonempty} \]