The Cantor Set

Recall that a perfect set \( E \) in a metric space \( X \) is such that every \( x \in E \) is a limit point of \( E \). Also recall that perfect sets are uncountable.

The Cantor set is an important example for several reasons. It is a perfect subset of \( \mathbb{R} \) that contains no interval. (As a corollary) it is also totally disconnected in the sense that its only connected subsets are points.

Much later you will see it again as an example of a set with "dimension" strictly between 0 and 1.
The middle third Cantor set is

\[ C = \{ x \in \{0,1,2\}^\mathbb{N} : x \text{ has no } 2\text{'s in its ternary expansion} \} \]

(although this classification isn't so useful)

The construction is inductive

Call \( C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \)

(removing the middle third of the interval)

Then \( C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{7}{9}, \frac{8}{9}] \cup [\frac{10}{9}, 1] \)

at each stage

\[ C_n = \bigcup_{i=1}^{3^n} I_i \]

of length \( 3^{-n} \).

\[ C_n < C_{n-1} < \ldots \]
Given $C_0$, define $C_n$ by as $(I_i^1 - (a_i, b_i)]$.

$$C_n = \bigcup_{i=1}^{2^n} \left( A_i^n, a_i^n + \frac{b_i^n - a_i^n}{3} \right) \cup \left( B_i^n - \frac{b_i^n - a_i^n}{3}, b_i^n \right)$$

Indeed, this is union of $2^{n+1}$ closed disjoint intervals of length

$$\frac{1}{3^n} \text{ length } I_i^n = 3^{-n+1}$$

Call $C = \bigcap_{n=1}^{\infty} C_n$ the middle $\frac{1}{3}$ Cantor set.

$C$ is compact and non-empty (Cantor intersection) of compact.

Further, no segment

$$\left( \frac{3b_n}{3^n}, \frac{3b_{n+2}}{3^n} \right)$$

intersects $C$.

(Since it is removed at stage $n$)
but every interval $I_{i12}$ has a segment of thin from contained in it. \[ I = [a, \infty) \]

Let $n \geq 3^m$.

Let $k$ be minimal in $Z$ suit.

\[ \frac{1}{2^{k+1}} \in (\alpha, \beta) \]

by minimality.

\[ \frac{3(k-1)+1}{3^n} \leq \alpha \]

\[ \frac{3k-1}{3^n} - \frac{2k-1}{3^{n+1}} \geq \alpha \leq \frac{1}{3^n-1} \]

so

\[ \frac{3(k+2)}{3^n} - \alpha \leq \frac{1}{3^n} + \frac{1}{3^{n+1}} = \frac{4}{3^n} \leq \frac{4(\beta - \alpha)}{\beta - \alpha} < \beta - \alpha \]
to show that $C$ is perfect we show that $C$ has no isolated point.

Let $x \in C$ and $A = (a, b)$ be an open interval containing $x$.

Since $x \in C$, for $n \in \mathbb{N}$ let $I_n$ be the interval of $C$ containing $x$.

For $n$ sufficiently large (i.e., $3^{-n} < \min \{x - a, b - x\}$), $I_n \subseteq A$.

Let $x_0$ be the other end point of $I_n$ which is not $x$.

Thus, for every open interval containing $x$, $I_n$ intersects $C$ at a pt other than $x$.

Hence $C$ is perfect.

Rem: $C$ has measure zero.