FINITE GROUPS WHOSE REAL IRREDUCIBLE REPRESENTATIONS HAVE UNIQUE DIMENSIONS

THOMAS BREUER, FRANK CALEGARI, SILVIO DOLFI, GABRIEL NAVARRO, AND PHAM HUU TIEP

Abstract. We determine the finite groups whose inequivalent real irreducible representations have different degrees.

1. Introduction

In this paper, we answer the following question: what are the finite groups whose real irreducible representations all have different degrees? Brauer's Problem 1 in his famous list of problems asks which finite-dimensional associative algebras are group algebras. For instance, if F is a subfield of the complex numbers, for which finite groups G we have that the algebra FG is a direct sum of pairwise non-isomorphic simple summands? If F is the field of complex numbers, then the complex group algebra is determined by the character degrees and their multiplicities. This multiplicity is of course related to the field of values of the character. In the problem mentioned above, that is, if all the irreducible complex characters of a finite group G occurs with multiplicity 1, then all the irreducible characters of G are necessarily rational valued. Using the Classification of Finite Groups and the main result from W. Feit and G. Seitz [FS89], it is easy to see that no non-trivial finite group has this property. (This was also noticed in [BCH92, Lemma 1].) There does not seem to be a proof of this result which avoids the Classification. In this paper, we are concerned with the real group algebra and the corresponding result for the inequivalent real irreducible representations. Perhaps somewhat surprisingly, we will show that exactly twelve finite non-trivial groups satisfy this property.

In fact, we work under a slightly more general hypothesis. If G is a finite group, let $\mathrm{Irr}(G)$ be the set of the irreducible complex characters of G, and recall that the Frobenius–Schur indicator of $\chi \in \mathrm{Irr}(G)$, $\nu(\chi)$, is 1 if χ can be afforded by a representation over the real numbers; it is -1, if it is real-valued but cannot be afforded by a real representation; and it is 0 if it is not real-valued.

THEOREM A. *Let G be a finite group. Suppose that whenever $\alpha, \beta \in \mathrm{Irr}(G)$ have the same degree and Frobenius–Schur indicator, then α and β are complex-conjugate.*

2010 *Mathematics Subject Classification.* Primary 20D20; Secondary 20C15.

Key words and phrases. Characters, Fields of Values, Degrees.

T. B. is supported by the German Research Foundation (DFG) – Project-ID 286237555 – within the SFB-TRR 195 Symbolic Tools in Mathematics and their Applications. F.C. was supported in part by NSF Grant DMS-2001097. S.D. was partially supported by INDAM-Gnsaga. The research of G. N. is supported by Ministerio de Ciencia e Innovación PID2022-137612NB-I00 funded by MCIN/AEI/ 10.13039/501100011033 and ERDF "A way of making Europe." P. H. T. gratefully acknowledges the support of the NSF (grants DMS-1840702 and DMS-2200850), the Simons Foundation, and the Joshua Barlaz Chair in Mathematics.

The authors are grateful to the referee for a careful reading and helpful comments on the paper.

(a) *If G is solvable, then G is a factor group of either $(C_2 \times C_2 \times C_2) \rtimes (C_7 \rtimes C_3)$, or $(C_p \times C_p) \rtimes \mathrm{SL}_2(3)$ for $p \in \{3, 5\}$.*

(b) *If G is non-solvable, then G is almost simple, and isomorphic to one of the groups in the following list*

$$\mathcal{L} = \{\mathsf{A}_8, \mathrm{SL}_3(2), \mathrm{M}_{11}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{SU}_3(3), \mathrm{McL}, \mathrm{Th}, \mathrm{SL}_2(8).3, \mathrm{O}_8^+(2).3\}. \tag{1.1}$$

Of these groups, McL is the unique group which admits real representations α, β of the same dimension (3520) but with distinct Frobenius–Schur indicator. Furthermore, the non-trivial groups in part (a) of Theorem A are isomorphic to C_3, $C_7 \rtimes C_3$, A_4, $\mathrm{SL}_2(3)$, $(C_3 \times C_3) \rtimes \mathrm{SL}_2(3)$, $(C_2 \times C_2 \times C_2) \rtimes (C_7 \rtimes C_3)$, and $(C_5 \times C_5) \rtimes \mathrm{SL}_2(3)$.

If G is a finite group, then recall that the degrees of the irreducible real representations are $\chi(1)$ if $\nu(\chi) = 1$, $2\chi(1)$ if $\nu(\chi) = -1$, and $2\chi(1)$ for the representatives of the non-real irreducible characters of G. (See, for instance, page 108 of [Ser77].) Using this, we easily derive the following.

COROLLARY B. *Let G be a finite group. Suppose that the degrees of the inequivalent real irreducible representations of G are all different.*

(a) *If G is solvable and non-trivial, then G is isomorphic to one of the following groups: C_3, A_4, $C_7 \rtimes C_3$, or $(C_2 \times C_2 \times C_2) \rtimes (C_7 \rtimes C_3)$.*

(b) *If G is non-solvable, then G is isomorphic to one of the following groups: A_8, M_{11}, M_{22}, M_{23}, M_{24}, McL, Th, or $\mathrm{SL}_2(8).3$.*

Conversely, if G is any of the groups listed in (a) *and* (b)*, then the degrees of the inequivalent real irreducible representations of G are all different.*

2. Solvable groups

For convenience, we say that a finite group G satisfies ***Hypothesis C*** if any two characters $\alpha, \beta \in \mathrm{Irr}(G)$ with the same Frobenius–Schur indicator and degree are complex-conjugate.

We remark that if G satisfies Hypothesis C, then every factor group of G also satisfies Hypothesis C, and moreover for every positive integer d, $\mathrm{Irr}(G)$ contains at most four characters of degree d. We also have $\mathbf{O}^2(G) = G$, since G has no non-trivial real-valued linear character. (Recall that $\mathbf{O}^p(G)$ is the smallest normal subgroup N of the finite group G such that G/N is a p-group.)

Lemma 2.1. *If $G > 1$ is a nilpotent group that satisfies Hypothesis C, then G is a cyclic group of order 3.*

Proof. Since $\mathbf{O}^2(G) = G$, we have that G has odd order. Hence, G/G' has odd order and every non-principal character $\lambda \in \mathrm{Irr}(G/G')$ is linear and non-real. So $\mathrm{Irr}(G/G') = \{1, \lambda, \overline{\lambda}\}$ and $G/G' = C_3$. (In this paper, G' denotes the derived subgroup of G, that is, the smallest normal subgroup with abelian factor group.) In particular, if $\Phi(G)$ is the Frattini subgroup of G, then we have that $G/\Phi(G)$ is cyclic, using that since G is nilpotent, then $G' \subseteq \Phi(G)$. Therefore G itself must be cyclic, and hence $G = C_3$. □

In this paper, we use the notation for characters in [Isa76]. Thus, if $H \leq G$ and $\tau \in \mathrm{Irr}(H)$, then τ^G denotes the induced character from H to G. Also recall that non-trivial irreducible characters of groups of odd order are non-real (by a theorem of Burnside, see Problem 3.16 of [Isa76]).

Lemma 2.2. *Assume that G satisfies Hypothesis C. If $N \trianglelefteq G$, G/N has odd order and cyclic Sylow subgroups, and $\tau \in \mathrm{Irr}(N)$ is non-real, then $\tau^G \in \mathrm{Irr}(G)$.*

Proof. Let G_τ be the inertia subgroup of τ in G. (That is, G_τ is the set of elements $g \in G$ such that $\tau^g = \tau$, where τ^g is the character of N defined by $\tau^g(n) = \tau(gng^{-1})$.) The assumption that the Sylow subgroups of G/N are cyclic implies that τ has an extension $\gamma \in \mathrm{Irr}(G_\tau)$ (by Corollary 11.22 of [Isa76].). If $G_\tau > N$ then, as $G_\tau/N \leq G/N$ is solvable, there exists a linear non-principal $\lambda \in \mathrm{Irr}(G_\tau/N)$. Hence, by the Clifford correspondence (Theorem 6.11 of [Isa76]), Gallagher's theorem (Corollary 6.17 of [Isa76]), and [IN12, Lemma 2.1], the induced characters γ^G and $(\lambda\gamma)^G$ of G are irreducible of the same Frobenius–Schur indicator and degree. Thus (since G satisfies Hypothesis C) they are complex-conjugate. This implies that τ and its complex conjugate $\overline{\tau}$ are conjugate by some element $g \in G$. Then $\tau^{g^2} = \tau$. Since G/N has odd order, τ is real, and this is a contradiction. $\square$

Lemma 2.3. *If G is a non-trivial group of odd order that satisfies Hypothesis C, then G is either cyclic of order 3 or a Frobenius group of order 21.*

Proof. Let $N = G'$, $K = N'$ and $L = K'$. We have that $G/N = C_3$. Let $1 \neq \lambda, \nu \in \mathrm{Irr}(N/K)$. Since G/N has prime order, then either λ extends to G or $G_\lambda = N$ and therefore λ^G is irreducible (by the Clifford correspondence). Since linear characters of G have N in their kernel, we then have that λ^G and $\nu^G \in \mathrm{Irr}(G)$ should be irreducible. They also have the same Frobenius–Schur indicator 0, because groups of odd order do not have non-trivial real-valued irreducible characters. So λ^G and ν^G are either equal or complex-conjugate. Thus λ is G-conjugate to ν or to $\bar{\nu}$, and it follows that the action of G/N on $\mathrm{Irr}(N/K) - \{1_N\}$ has two orbits of size 3. (Notice that ν and $\bar{\nu}$ cannot be G-conjugate, since G has odd order.) Thus N/K has order 7, and $G/K = C_7 \rtimes C_3$. Suppose that $K > L$. Let $1 \neq \lambda, \nu \in \mathrm{Irr}(K/L)$. By Lemma 2.2, we have that $\lambda^G, \nu^G \in \mathrm{Irr}(G)$. Since these characters have the same Frobenius–Schur indicator and degree, we conclude that they are equal or complex-conjugate. Thus, again, λ is G-conjugate to ν or to $\bar{\nu}$. Since $G_\lambda = K$ (using that λ^G is irreducible and Problem 6.1 of [Isa76]), we have that the G-orbit of λ has size 21. Therefore K/L has order 43. Thus K/L is cyclic of order 43. But then $(G/L)/\mathbf{C}_{G/L}(K/L)$ is abelian, because it is isomorphic to a subgroup of the automorphism group of a cyclic group. Then $N/L \subseteq \mathbf{C}_{G/L}(K/L)$, and therefore K/L is in the center of N/L. This is impossible since there are characters of K/L that induce irreducibly to G/L. $\square$

We write $\mathrm{Irr}_+(G) = \{\chi \in \mathrm{Irr}(G) \mid \nu(\chi) = 1\}$, and denote by $A\Gamma(2^3)$ the affine group $C_2^3 \rtimes (C_7 \rtimes C_3)$.

Proposition 2.4. *If G satisfies Hypothesis C and has a normal Sylow 2-subgroup, then it is isomorphic to a factor group of either $\mathrm{SL}_2(3) = Q_8 \rtimes C_3$ or $A\Gamma(2^3)$.*

Proof. Let N be the normal Sylow 2-subgroup of G. By Lemma 2.3, $G/N \cong C_3$ or $G/N \cong C_7 \rtimes C_3$. We can assume $N \neq 1$.

Assume first $G/N \cong C_3$. Then $N = G'$, since $\mathbf{O}^2(G) = G$. If $\lambda \in \mathrm{Irr}(N)$ is linear of order 2, then $\lambda^G \in \mathrm{Irr}_+(G)$, so by Hypothesis C and Clifford's Theorem there is a unique G-orbit of such characters. Hence, $N/\Phi(N) \cong C_2 \times C_2$ is an irreducible T-module, where $T \cong C_3$ is a Sylow 3-subgroup of G. This implies that N/N' is an indecomposable T-module, so it is homocyclic. For $\lambda \in \mathrm{Irr}(N/N')$ with $o(\lambda) > 2$, it follows from Lemma 2.2 that λ^G is an irreducible character of G and it is non-real (by [IN12, Lemma 2.1]), so there are at most two T-orbits of characters of order > 2 in $\mathrm{Irr}(N/N')$. Since N/N' is homocyclic and $C_{2^n} \times C_{2^n}$ has $2^{2n} - 2^2 \geq 12$ linear characters of order > 2 when $n > 1$, this implies $N/N' \cong C_2 \times C_2$ and

$G/N' \cong \mathsf{A}_4$. If $N' \neq 1$, then N is either dihedral, semidihedral or a generalized quaternion group (by [Hup67, Satz III.11.9]) and, observing that $\mathrm{Aut}(N)$ is not a 2-group, we conclude that $N \cong Q_8$ and $G \cong \mathrm{SL}_2(3)$.

Assume now that $G/N \cong C_7 \rtimes C_3$ and let $H = TR$ be a complement of N in G, where $T \cong C_7$ and $R \cong C_3$ are Sylow subgroups of G. By considering a suitable factor group of G, we may assume first that N is minimal normal in G. We observe that T does not centralize N, as otherwise (recalling that $\mathbf{O}^2(G) = G$) N is a faithful irreducible R-module, so $N \cong C_2 \times C_2$ and G would be a Frobenius group with kernel $N \times T$, so G would have 9 irreducible characters of degree 3, a contradiction. By [Isa76, Theorem 15.16] (with $G = H$, $V = N$, $N = T$, $H = R$, and $H_0 = 1$), we have $\dim N = \dim \mathbf{C}_N(1) = 3 \dim \mathbf{C}_N(R)$, and thus there exists a non-trivial element $x \in \mathbf{C}_N(R)$, so by minimality $N = \langle x \rangle^G = \langle x \rangle^T \cong C_2 \times C_2 \times C_2$ and $G \cong A\Gamma(2^3)$. (Alternatively, the faithful irreducible representations of G/N over $\mathbb{F}_2$ all have dimension 3 and are unique up to automorphisms of G/N.)

In order to conclude the proof, we show that N is minimal normal in G. We first observe that $N/\Phi(N)$ is an irreducible H-module: otherwise, since it is a completely reducible H-module, by the previous paragraph there exist distinct normal subgroups K_1 and K_2 of G such that $G/K_i \cong A\Gamma(2^3)$, for $i = 1, 2$, so, as $A\Gamma(2^3)$ has two irreducible non-real characters of degree 7, there would be at least four of them in G, against Hypothesis C.

As $N/\Phi(N)$ is irreducible, then N/N' is an indecomposable H-module and, using Lemma 2.2 as in the second paragraph of this proof, Hypothesis C implies that $N' = \Phi(N)$. We now show that $K = N'$ is trivial. Working by contradiction, we may assume (by taking a suitable quotient of G) that K is minimal normal in G, so $K = \mathbf{Z}(N)$. For $1 \neq \mu \in \mathrm{Irr}(K)$, by [Wol78, Lemma 2.2] there is a unique subgroup U_μ with $K \leq U_\mu \leq N$ such that μ extends to $\lambda \in \mathrm{Irr}(U_\mu)$ and λ is fully ramified in N/U_μ (for a definition of fully ramified characters see [Isa76, Problem 6.3]). As μ does not extend to N and $|N/U_\mu|$ is a square (using [Isa76, Theorem 6.18(b)]), it follows $|U_\mu/K| = 2$. So, $\mathrm{Irr}(N|\mu) = \{\theta_1, \theta_2\}$ has order 2, $\theta_i(1) = 2$ and $G_{\theta_i} = G_\mu$ for $i = 1, 2$. Note that K is a faithful H-module, otherwise H_μ has order divisible by 7 for some $1 \neq \mu \in \mathrm{Irr}(K)$, and this gives more than 4 irreducible characters of G with the same degree. In particular, we see that $T \in \mathrm{Syl}_7(H)$ acts fixed point freely on K. An application of [Isa76, Theorem 15.16] to the action of H on the dual module K^* (as $\mathbf{C}_{K^*}(T) = \mathbf{C}_K(T) = 1$) yields the existence of a non-principal irreducible character μ of K such that $R \leq H_\mu$. So, $G_{\theta_i} = NR$ for $\theta_i \in \mathrm{Irr}(N|\mu)$, $i = 1, 2$, and hence G has at least six irreducible characters of degree 14, a contradiction. Hence $K = 1$ and and $G \cong A\Gamma(2^3)$. □

In the following, we denote by $\mathbf{O}^{2'}(G)$ the smallest normal subgroup of G with odd index in G, and by $\mathbf{O}_2(G)$ the largest normal subgroup of G having order a power of 2.

Theorem 2.5. *Let G be a solvable group that satisfies Hypothesis C. Then G is isomorphic to a factor group of either $A\Gamma(2^3)$ or $(C_p \times C_p) \rtimes \mathrm{SL}_2(3)$ for $p \in \{3, 5\}$.*

Proof. Let $N = \mathbf{O}^{2'}(G)$ and $K = \mathbf{O}^2(N)$. If $K = N$, then N has no quotients of order 2 or of odd order > 1; since N is assumed to be solvable, this implies that N is trivial and we are done by Lemma 2.3; hence we can assume $K < N$. By Proposition 2.4, we deduce that G/K is isomorphic to either $A\Gamma(2^3)$, A_4 or $\mathrm{SL}_2(3)$, and hence we can assume that $K \neq 1$.

(I.) Suppose first $G/K \cong A\Gamma(2^3)$ and, working by contradiction, that $K \neq 1$. By considering a suitable factor group of G, we can assume that K is a minimal normal subgroup of G. Let $|K| = p^n$, for an odd prime p. We claim that K is the Fitting subgroup $\mathbf{F}(G)$. Otherwise, $N \leq \mathbf{F}(G)$, because N/K is the only minimal normal subgroup of G/K. But then N would be nilpotent and thus isomorphic to a direct product of K and $\mathbf{O}_2(N) \cong N/K$, and $G/\mathbf{O}_2(N)$

would be a group of odd order $21|N|$ that satisfies Hypothesis C, contradicting Lemma 2.3. Let $M \in \mathrm{Syl}_2(N)$ and $H = \mathbf{N}_G(M)$. Then H is a complement of K in G, as $G = KH$ by the Frattini argument and $K \cap H = \mathbf{C}_K(M) = \mathbf{Z}(N) \cap K \trianglelefteq G$, so $K \cap H = 1$ as K is minimal normal in G. Let $T \cong C_7$ be a Sylow 7-subgroup of H. Then MT is a Frobenius group and hence $\mathbf{C}_K(T) \neq 1$ by [Isa76, Theorem 15.16]. Thus, by [Isa76, Theorem 6.32] there exists a non-principal $\mu \in \mathrm{Irr}(K)$ such that $T \leq H_\mu$. Note that the inertia subgroup N_μ is a proper subgroup of N, since μ extends to N_μ by [Isa76, Corollary 6.28] and μ has no estension $\nu \in \mathrm{Irr}(N)$, as otherwise $K = N' \leq \ker(\nu_K) = \ker(\mu)$ against $\mu \neq 1_K$. Hence, $N_\mu = K$ because N_μ is T-invariant and N/K is an irreducible T-module. Thus $\theta = \mu^N \in \mathrm{Irr}(N)$. We show that θ is non-real. In fact, otherwise there exists an element $x \in M$ (as M is a Sylow 2-subgroup of G and $KM = N$ contains every 2-element of G) such that $\mu^x = \overline{\mu} = \mu^{-1}$. So, considering the action of H on the dual module K^*, $\mathbf{N}_M(\langle \mu \rangle)$ is a non-trivial and T-invariant subgroup of M. Hence $\mathbf{N}_M(\langle \mu \rangle) = M$, since M is an irreducible T-module. But then $\mathbf{C}_M(\langle \mu \rangle)$ is a T-invariant subgroup of index 2 in M, a contradiction. As $\theta \in \mathrm{Irr}(N)$ is non-real, then $\theta^G \in \mathrm{Irr}(G)$ by Lemma 2.2, which is a contradiction as $T \leq G_\mu \leq I_G(\theta)$. Therefore, $K = 1$ if $G/K \cong A\Gamma(2^3)$.

(II.) We now assume $G/K \cong \mathsf{A}_4$ and, again, that K is a minimal normal subgroup of G, $|K|$ odd. Let $M \in \mathrm{Syl}_2(G)$. As above, $K = \mathbf{F}(G)$, as otherwise $N = M \times K$, with $M \trianglelefteq G$, against Lemma 2.4. Since $N = KM$ and $\mathbf{C}_K(N) \trianglelefteq G$, by the minimality of K we deduce that $\mathbf{C}_K(M) = \mathbf{C}_K(N) = 1$. By the Frattini argument $G = KH$ where $H = \mathbf{N}_G(M)$ and $K \cap H = \mathbf{C}_K(M) = 1$, so $H = MR \cong \mathsf{A}_4$ is a complement of K in G, where $R \in \mathrm{Syl}_3(G)$. Then, considering the action of H on the dual module K^*, $\mathbf{C}_{K^*}(M) = 1$ and by [Isa76, Theorem 15.16] there exists a non-principal and R-invariant $\mu \in K^*$ and, as M_μ is an R-invariant and proper subgroup of M (as $\mathbf{C}_{K^*}(M) = 1$), we deduce that $M_\mu = 1$ and hence $\theta = \mu^N \in \mathrm{Irr}(N)$. As in the previous paragraph, one shows that θ is non-real, giving a contradiction by Lemma 2.2.

(III.) Finally, we assume $G/K \cong \mathrm{SL}_2(3)$ and we prove, by induction on $|G|$, that either $G \cong G_1 = (C_3 \times C_3) \rtimes \mathrm{SL}_2(3)$ or $G \cong G_2 = (C_5 \times C_5) \rtimes \mathrm{SL}_2(3)$.

To start with, we prove that K is nilpotent. Working by contradiction, we assume that $F = \mathbf{F}(K) < K$ and, by considering a suitable factor group of G, that F is minimal normal in G. By induction, $G/F \cong G_i$ for some $i \in \{1, 2\}$. Write $|F| = q^m$, where q is a prime number, and observe that $q \neq p$, where $p = 3$ if $i = 1$, and $p = 5$ if $i = 2$. For $M \in \mathrm{Syl}_p(K)$, $\mathbf{C}_F(M) = \mathbf{C}_F(K) = 1$ by the minimality of F, and the Frattini argument implies that $H = \mathbf{N}_G(M)$ is a complement of F in G. Let $L = N \cap H$; then $L \cong (C_p \times C_p) \rtimes Q_8$ has index 3 in H. In order to get a contradiction, we will show that then L has at least 14 regular orbits on the dual module $V = F^*$. In fact, the regular L-orbits are permuted by the action of $\overline{H} = H/L \cong C_3$, and the ones that are not fixed by $\overline{H}$ give, in groups of three, regular orbits of H on V. If, instead, Δ is a regular L-orbit in V such that Δ is $\overline{H}$-invariant, then Δ is an H-orbit of size $|L|$. Thus, for $\lambda \in \Delta$, $|H_\lambda| = 3$ and by Clifford Correspondence $\mathrm{Irr}(G|\lambda)$ contains three irreducible characters of degree $|L|$. Hence, if L has at least $14 = 4 \times 3 + 2$ regular orbits on V, then G has more than four irreducible characters either of degree $|H|$ or of degree $|L|$, against Hypothesis C.

Assume first $p = 3$: so L has four subgroups of order 3 and nine subgroups of order 2. These are the only subgroups of prime order of L and hence every element in F that is not centralized by any of them is in a regular L-orbit. If $X \leq L$ and $|X| = 3$, then $|\mathbf{C}_V(X)| \leq q^{m/2}$ because two distinct subgroups X_1, X_2 of order 3 of L generate $K \cap H \cong C_3 \times C_3$, so $\mathbf{C}_V(X_1) \cap \mathbf{C}_V(X_2) = \mathbf{C}_V(K \cap H) = \mathbf{C}_F(K) = 1$. If $X \leq L$ and $|X| = 2$, then $|\mathbf{C}_V(X)| = q^{m/2}$

by [Isa76, Theorem 15.16]. Moreover, in characteristic $q \neq 3$, all faithful absolutely irreducible representations of $H \cong G_1$ have degree 8, as one can check by looking at its (ordinary and) Brauer character tables, so m is a multiple of 8. Now, the function

$$g(q,m) = (q^m - 1) - (9+4) \cdot (q^{m/2} - 1) - 14|L|$$

is > 0 for all primes q and $m \geq 8$, except for the pair $(q,m) = (2,8)$. Using [GAP21], one checks that H has a unique faithful (absolutely) irreducible module F_0 of dimension 8 over $\mathbb{F}_2$ and that the semidirect product $G_0 = F_0 H$ has 6 irreducible characters of degree 72, so does not satisfy Hypothesis C. Hence, $p \neq 3$.

Finally, if $p = 5$ then L has 6 subgroups of order 5 and 25 subgroups of order 2. As in the previous paragraph, one checks that for $X \leq L$, $|X| \in \{2,5\}$, $|\mathbf{C}_F(X)| \leq |F|^{\frac{1}{2}}$. Moreover, from the Brauer character tables of $H \cong G_2$ it turns out that, for every characteristic $q \neq 5$, H has just one faithful absolutely irreducible representation, which has dimension 24. So m is a multiple of 24, and one easily checks that

$$h(q,m) = (q^m - 1) - (6+25) \cdot (q^{m/2} - 1) - 14|L|$$

is positive for all primes q and integers $m \geq 24$.

Hence, we have proved that K is nilpotent. It follows that $|K|$ is odd and that $K = \mathbf{F}(G)$, as otherwise $\mathbf{O}_2(G) \neq 1$ and factor group $G/\mathbf{O}_2(G)$ contradicts the inductive hypothesis. We now prove that K is abelian. Working again by contradiction, we assume that $X = K' \neq 1$. By induction, X is a minimal normal subgroup of G and G/X is isomorphic to either G_1 or G_2. Hence, K/X is a minimal normal subgroup of G/X and it follows that $X = \mathbf{Z}(K) = \mathbf{\Phi}(K)$. Since $|K/X| \in \{3^2, 5^2\}$, we deduce that K is an extraspecial group of order p^3, $p \in \{3,5\}$. Moreover, $G/\mathbf{C}_G(X)$ is an abelian group of order dividing $|G/G'| = 3$, so $X \leq \mathbf{Z}(G)$. Thus, the $p-1$ irreducible characters of degree p of K extend to G by [Isa76, Corollary 11.31], as G/K has either cyclic or quaternion Sylow subgroups, and hence by Gallagher's theorem ([Isa76, Corollary 6.17]) G has $3(p-1) \geq 6$ irreducible characters of degree p, a contradiction. Thus, K is abelian.

Let $Q \cong Q_8$ be a Sylow 2-subgroup of G and $Z = \mathbf{Z}(Q)$. As K is an abelian of odd order, then $K = \mathbf{C}_K(Z) \times [K,Z]$. Since $KZ \trianglelefteq G$, $[K,Z] = [K,KZ] \trianglelefteq G$ and then $L = [K,Z]Z \trianglelefteq G$, since $L/[K,Z]$ is the Sylow 2-subgroup of the abelian normal subgroup $KZ/[K,Z]$ of $G/[K,Z]$. So $\overline{G} = G/L$ satisfies Hypothesis C and $\overline{G}/\overline{K} \cong \mathsf{A}_4$, with $\overline{K} = \mathbf{O}^2(\mathbf{O}^{2'}(\overline{G}))$, and hence by case (II) $\overline{K} = 1$, which gives $\mathbf{C}_K(Z) = 1$. So, the involution $z \in Z$ acts by inverting all elements of K and, in particular, $\mathbf{C}_K(Q) = 1$.

Thus, by the Frattini argument $H = \mathbf{N}_G(Q)$ is a complement of K in G. Note that $H = QR \cong \mathrm{SL}_2(3)$, where $|R| = 3$. If $\mu \in \mathrm{Irr}(K)$ is such that $H_\mu = 1$ (so $G_\mu = K$), then $\chi = \mu^G \in \mathrm{Irr}(G)$ and χ is a real character as $\overline{\mu} = \mu^{-1} = \mu^z$. By [Isa76, Lemma 10.8], the Schur index $m_{\mathbb{R}}(\chi)$ divides $\mu(1) = 1$ and hence $\chi \in \mathrm{Irr}_+(G)$ by [Isa76, Theorem 10.17]. Therefore, by Hypothesis C there is at most one regular orbit of G/K on $\mathrm{Irr}(K)$.

Next, we show that K is a group of prime power order. Otherwise, the inductive hypothesis applied to $G/\mathbf{O}_p(K)$ for each prime divisor p of $|K|$ yields $G \cong (A \times B) \rtimes \mathrm{SL}_2(3)$, with $A \cong C_3 \times C_3$ and $B \cong C_5 \times C_5$. Moreover, G/K acts regularly on $\mathrm{Irr}(B) \smallsetminus \{1_B\}$ and hence, for any $\alpha \in \mathrm{Irr}(A)$ and any $\beta \in \mathrm{Irr}(B)$, $\beta \neq 1_B$, $\alpha \times \beta$ has stabilizer $A \times B = K$ in G. Thus G/K has 9 regular orbits on $\mathrm{Irr}(K)$, a contradiction by the previous paragraph.

Thus, K is an abelian p-group for some prime $p \neq 2$. Let $|K| = p^n$, for some positive integer n, and let $V = K^*$ be the dual H-module. Note that, as $K = \mathbf{F}(G)$, both K and V

are faithful H-modules. Next, we show that

$$(p, n) \in \{(3, 2), (5, 2), (7, 2)\}.$$

Let $\mathrm{Syl}_3(H) = \{R_i | 1 \leq i \leq 4\}$ and observe that $\mathbf{C}_V(R_i) \cap \mathbf{C}_V(R_j) = 1$ for $i \neq j$, because $\langle R_i, R_j \rangle = H$ and $\mathbf{C}_V(R_i) \cap \mathbf{C}_V(R_j) = \mathbf{C}_V(\langle R_i, R_j \rangle) = \mathbf{C}_K(H) \leq \mathbf{C}_K(Q) = 1$. Thus, $|\mathbf{C}_V(R_i)| \leq p^{[n/2]}$. Define $f(p, n) = p^n - 4p^{[n/2]} - 21$ and observe that, writing $V^\# = V \smallsetminus \{1_K\}$,

$$|V^\#| - |\bigcup_{i=1}^{4} \mathbf{C}_{V^\#}(R_i)| = (p^n - 1) - 4(p^a - 1) \tag{2.1}$$

is larger than $|H| = 24$ if $f(n, p) > 0$. So, if $f(p, n) > 0$ then H has at least two regular orbits on V and hence, as observed above, G does not satisfy Hypothesis C. We also observe that n is even, as every indecomposable summand of V as a Q-module is a homocyclic group of even rank because $\mathbf{C}_V(Q) = \mathbf{C}_K(Q) = 1$. In particular, $n \geq 2$.

One can check that $f(p, n) > 0$ for all pairs (p, n) with $p > 7$ and $n \geq 2$, or such that $n \geq 4$ (and $p \neq 2$), thus proving the claim. So, considering that K is non-cyclic because H acts faithfully on K, we deduce that K is an elementary abelian group, with $|K| \in \{3^2, 5^2, 7^2\}$.

If $|K| = 7^2$, then G is either $\mathtt{SmallGroup}(1176, 214)$ or $\mathtt{SmallGroup}(1176, 215)$ (in the GAP library of small groups [BEOH23]). In fact, H has three isomorphism classes of faithful (irreducible) modules K of dimension 2 over $GF(7)$ and the corresponding semidirect products $G = K \rtimes H$ belong to the two stated isomorphism types. In the first case, H has three orbits of size 8 on K^*, so by Clifford theory G has nine irreducible characters of degree 8. In the second case, H acts semiregularly on K^*, so it has two regular orbits on K^*. Hence, both these groups do not satisfy Hypothesis C.

Finally, H has exactly one isomorphisme type of faithful (irreducible) modules of dimension 2 over both $GF(3)$ and $GF(5)$. The corresponding semidirect products $G = K \rtimes H$ are, respectively, isomorphic to G_1 and G_2, concluding the proof.

We also observe that both $G_1 = \mathtt{SmallGroup}(216, 153)$ and $G_2 = \mathtt{SmallGroup}(600, 150)$ (in [BEOH23]) satisfy Hypothesis C. □

3. Almost Simple groups

Theorem 3.1. *If G is an almost simple group and G satisfies Hypothesis C, then G is one of the groups in the list $\mathcal{L}$ defined in* (1.1).

Proof. (a) Let S denote the socle of G, so that $S \lhd G \leq \mathrm{Aut}(S)$. First we consider the case $S = \mathsf{A}_n$ with $n \geq 5$. Since the cases $5 \leq n \leq 12$ can be checked directly using [CCN+85], we will assume that $n \geq 13$, and hence $G = S$ since S_n does not satisfy Hypothesis C. To rule out such groups G, it suffices to find a partition $\lambda \vdash n$ which is self-associate, such that the Young diagram of λ has k nodes on the main diagonal and $k \equiv n \pmod 4$. Indeed, in this case, the character $\chi \in \mathrm{Irr}(\mathsf{S}_n)$ labeled by λ splits upon restriction to A_n into the sum $\chi^+ + \chi^-$ of two real-valued irreducible characters of A_n, which are S_n-conjugate and hence of the same degree and same Frobenius–Schur indicator, see [JK81, Theorem 2.5.13]. Now, if $n \equiv 1 \pmod 4$, we take $\lambda = ((n+1)/2, 1^{(n-1)/2})$, the symmetric hook partition of n. If $n \equiv 2 \pmod 4$, take $\lambda = (n/2, 2, 1^{n/2-2})$, the symmetric hook partition of $n-1$ augmented by a node in the second row. If $n \equiv 3 \pmod 4$, take $\lambda = ((n-3)/2, 3, 3, 1^{(n-9)/2})$, the symmetric hook partition of $n-4$ augmented by a 2×2-square in the second and third rows. If $n \equiv 0 \pmod 4$, take $\lambda = ((n-8)/2, 4, 4, 4, 1^{(n-16)/2})$, the symmetric hook partition of $n-9$ augmented by a 3×3-square in the second, third, and fourth rows.

(b) Checking the almost simple groups with sporadic socle S using [CCN+85], we may now assume that S is a simple group of Lie type. Next we observe that $[\mathbb{Q}(\chi):\mathbb{Q}] \leq 2$ for any $\chi \in \mathrm{Irr}(G)$. Indeed, the Γ-orbit of χ has length $|\Gamma|$, where $\Gamma := \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. Clearly, any Galois conjugate of χ has the same degree and Frobenius–Schur indicator as χ. Hence Hypothesis C implies that $2 \geq |\Gamma| = [\mathbb{Q}(\chi):\mathbb{Q}]$.

Thus G is a *quadratic-rational group*, in the sense of [Tre17]. Therefore, all the possibilities for S are listed in [Tre17, Theorem 1.2]; furthermore, G/S is a solvable group with no normal subgroup of index 2 or 5 (by Hypothesis C applied to G/S). All such groups G can be checked using GAP. □

4. Modules and the Almost Simple Groups

Theorem 4.1. *Suppose that $\tilde{G}$ is a finite group having a unique minimal normal subgroup N such that N is an irreducible faithful module for $G = \tilde{G}/N$ and G is one of the groups in the list $\mathcal{L}$ defined in* (1.1). *Then $\tilde{G}$ does not satisfy Hypothesis C.*

Proof. The proof is computational. In the computations, we used the character tables and tables of marks from the GAP packages [Bre24] and [MNP24], respectively. The matrix representations were taken from the Atlas of Group Representations [WWT+] or derived from representations available there. Cohomology groups and group extensions were computed using implementations in [Hol23, GAP21, OSC24]. Permutation representations of small degrees of the group extensions were obtained using ideas from [Hul01], and [BCP97] was used to compute the character tables.

By assumption, N is an irreducible faithful G-module of dimension n over a prime field $\mathbb{F} = \mathbb{F}_p$ with p elements. Viewing $\mathrm{Irr}(N)$ as a vector space over $\mathbb{F}$, we can identify it with the dual module $M = N^*$. By Clifford theory, each regular orbit of G on M yields an irreducible character of degree $|G|$ of $\tilde{G}$, see [Isa76, Theorem 6.11]. Thus we are done if we show that G has at least five regular orbits on M. We use the techniques from [FOS16] and [FMOW19] to derive lower bounds for the number $R(G, M)$ of regular orbits, as follows.

By construction, M is an irreducible faithful G-module of dimension n over $\mathbb{F}$. Let P be a set that contains one generator for each cyclic subgroup of prime order in G. Then

$$M = \left\{v \in M; |\operatorname{Stab}_G(v)| = 1\right\} \cup \left(\bigcup_{g\in P} \operatorname{Fix}_M(g)\right)$$

implies

$$|M| \leq |G| \cdot R(G, M) + \sum_{g\in P} |\operatorname{Fix}_M(g)|.$$

Let T denote the socle of G. For a nonidentity element $g \in G$, define $r(g)$ to be the minimal number of T-conjugates of g that generate $\langle T, g\rangle$, and set

$$r(G) = \max\{r(g); g \in G, |g| \neq 1\}.$$

Then [FOS16, Lemma 3.4] yields

$$\dim_{\mathbb{F}}(\operatorname{Fix}_M(g)) \leq \dim_{\mathbb{F}}(M)(1 - 1/r(g)).$$

The values $r(g)$ for the groups listed in the theorem are given by [FMOW19, Theorem 1.3] for sporadic simple groups, and have been computed directly for the other groups in question; they can be read off from Table 1, in the following sense: We always have $r(g) \geq 3$ if g has order 2, and $r(g) \geq 2$ otherwise; Table 1 lists all cases where the inequality is strict.

G	$r(g)=3$	$r(g)=4$	$r(g)=5$
A_8		$2A, 2B, 3A$	
M_{22}		$2B$	
$\mathrm{SU}_3(3)$	$3A$	$2A$	
McL	$3A$		
$\mathrm{O}_8^+(2).3$	$3D, 3E, 3F$	$3ABC$	$2A, 2BCD$

TABLE 1. Exceptional values of $r(g)$.

It follows that

$$|G| \cdot R(G,M) \geq p^n - \sum_{g \in P} p^{n(1-1/r(g))} \geq p^{n(1-1/r(G))} \left(p^{n/r(G)} - |P| \right).$$

Thus $R(G,M) \geq 5$ holds except for only finitely many pairs (p,n).

We define $D(G)$ to be the smallest integer n such that

$$5|G| \geq 2^{n(1-1/r(G))} \left(2^{n/r(G)} - |P| \right)$$

holds. Then $R(G,M) \geq 5$ holds for any G-module M of dimension larger than $D(G)$ over a finite prime field.

The Brauer characters of those irreducible G-modules M for which this crude estimate does not suffice are listed in [JLPW95] and (if the characteristic p does not divide $|G|$) in [CCN$^+$85]. For each p'-element g, we can compute the exact value of $\dim_{\mathbb{F}}(\mathrm{Fix}_M(g))$ from the Brauer character of M, which yields better estimates for $R(G,M)$ and rules out many cases. The modules for which this does not establish at least five regular orbits are listed in Table 2. Direct computations with the table of marks of G and with matrices that describe the G-action on M yield the orbit lengths. Finally, if this is not sufficient, we compute the possible extensions $\tilde{G}$ of G by $N \cong M^*$, and verify directly that these groups $\tilde{G}$ do not satisfy Hypothesis C.

For example, let G be the Thompson group Th. We have

$$r(G) = 3 \text{ and } |P| = 675\,176\,077\,846\,831,$$

thus $p^{n/r(G)} - |P| > 0$ if $n \geq 148$ holds, and $2^{2n/3}(2^{n/3} - |P|) > 5|G|$ in this case; hence $D(G) = 148$. Since the minimal degree of a nontrivial irreducible representation of G in any characteristic is 248 (see [Jan05]), we have shown that the claim holds for the group Th.

For the group $G = \mathrm{O}_8^+(2).3$, we compute $D(G) = 113$, thus the five modules listed in Table 2 must be considered.

- The 24-dimensional representation over $\mathbb{F}_2$ is induced from $\mathrm{O}_8^+(2)$, thus we consider the restriction of the possible group extensions (there are two, a split and a non-split one) to the index 3 subgroup $2^{24}.\mathrm{O}_8^+(2)$. The 24-dimensional module splits as a direct sum of three 8-dimensional ones, thus there are three factor groups $2^8.\mathrm{O}_8^+(2)$, which are permuted cyclically by the automorphism of order three. The groups $2^8.\mathrm{O}_8^+(2)$ (there are two, a split and a non-split one) have the character degree 2835 with multiplicity 6, which yields the degree $3 \cdot 2835$ with multiplicity at least 6 for $2^{24}.\mathrm{O}_8^+(2).3$.
- The restriction of the 26-dimensional representation over $\mathbb{F}_2$ to $\mathrm{O}_8^+(2)$ has a unique orbit of length 1575 on the module. Hence also G has this orbit. We compute that there is only one possible extension of G with the module, which is split, thus it is

G	$D(G)$	p	M	$R_1(R_2)$	H^2	deg.
A_8	45	2	4a		1	$+105^3$
			4b		1	$+105^3$
			6a		0	35^5
			14a		2	210^5
			20a	37		
			20b	37		
		3	7a		1	28^5
			13a	23		
		5	7a		0	28^9
		7	7a	(15)		
		11	7a	240		
		13	7a	1122		
$\mathrm{SL}_3(2)$	18	2	3a		1	$+21^2$
			3b		1	$+21^2$
			8a		0	24^7
		3	6a		0	21^{24}
			3ab		0	42^{16}
		7	3a		1	24^{14}
M_{11}	30	2	10a		1	55^5
		3	5a		0	$+110^2$
			5b		0	220^5
			10a		0	55^9
			10b	5		
			10c	5		
M_{22}	64	2	10a			$+1155^2$
			10b			$+1155^2$
		3	21a	20241		
M_{23}	59	2	11a		1	$+253^2$
			11b		0	$+253^3$

G	$D(G)$	p	M	$R_1(R_2)$	H^2	deg.
M_{24}	68	2	11a		0	$+239085^4$
			11b		1	$+239085^2$
			44a	69255		
			44b	69173		
		3	22a	34		
$\mathrm{SU}_3(3)$	38	2	6a		1	$189^6, +378^2$
			14a		0	252^{13}
		3	3ab		0	224^9
			6ab	40		
			7a		2	$672^9, 756^8$
		5	6a		0	756^8
			7b		0	108^8
			7c		0	108^8
		7	6a	16		
McL	77	2	22a		1	311850^6
		3	21a	(10)		
Th	148		(none)			
$\mathrm{SL}_2(8).3$	24	2	6a		1	63^6
			12a		0	63^6
			8a		0	36^6
			8bc	34		
		3	7a		2	56^9
$\mathrm{O}_8^+(2).3$	113	2	24a			
			26a			
			26bc			
		3	28a			
		3	48a			

Table 2. Modules to be considered.

enough to establish a big enough degree multiplicity among the ordinary irreducible characters of the point stabilizer, which is an involution centralizer of order 331776. This subgroup has the irreducible degree 16 with multiplicity 6, which yields multiplicity at least 6 for the degree $1575 \cdot 16$ in $2^{26}.\mathrm{O}_8^+(2).3$.

- The restriction of the 52-dimensional representation over $\mathbb{F}_2$ to $\mathrm{O}_8^+(2)$ is twice the 26-dimensional representation which we dealt with above. This restriction has 25493462 regular orbits, which is enough for $\mathrm{O}_8^+(2).3$.
- The restrictions of the G-representations of the degrees 28 and 48 over $\mathbb{F}_3$ to $\mathrm{O}_8^+(2)$ have 127877 and 457929531422105 regular orbits, which yields huge degree multiplicities also for $\mathrm{O}_8^+(2).3$.

In this case, we chose to work with restrictions to $\mathrm{O}_8^+(2)$ since the table of marks of this simple subgroup is available, but the table of marks for $\mathrm{O}_8^+(2).3$ is not.

For the other groups G in question, Table 2 describes the arguments that exclude the modules, as follows.

- The column with header p lists the characteristics for which at least one irreducible $\mathbb{F}G$-module M exists such that the character-theoretic lower bound for the number of regular orbits is less than 5.
- The column with header M lists the irreducible modules to be considered. Names consisting of an integer n followed by one letter `a`, `b`, etc. denote the first, second,

etc. absolutely irreducible module of dimension n, names involving several letters denote modules that decompose into a sum of the corresponding absolutely irreducible modules of dimension n over an extension field; for example `3ab` means a module that is irreducible over the prime field with p elements, of dimension 6, and decomposes into two absolutely irreducible modules over the field with p^2 elements.

- The column with header $R_1(R_2)$ contains the number of regular orbits, provided this number is at least 5, or in brackets the number of orbits of length $|G|/2$, provided this number is at least 9. Note that each orbit of length $|G|/2$ corresponds to an inertia subgroup which has only the character degrees 1 and 2, hence 9 orbits yield one character degree with multiplicity at least 5.

  The values in this column have been computed from the known table of marks of G, together with matrix generators describing M.
- If the column $R_1(R_2)$ does not exclude the module in question then the column with header H^2 contains the dimension of the corresponding second cohomology group that describes the possible group extensions, and the column with header deg. contains a description of character degrees that exclude the module. In this column deg., the entry d^n means that the character degree d occurs with multiplicity n, and the entry $+d^n$ means that degree d occurs with multiplicity n for characters with indicator $+$.

$\square$

5. Non-Solvable groups

Lemma 5.1. *Let S be a non-abelian simple group and $A = \mathrm{Aut}(S)$. Then, there exists a non-principal character $\phi \in \mathrm{Irr}_+(S)$ having an extension $\hat{\phi} \in \mathrm{Irr}_+(A)$.*

Proof. If S is a simple group of Lie type, not isomorphic to ${}^2F_4(2)'$, we let ϕ be the Steinberg character of S. It is known ([Fei93, Theorem B]) that ϕ has an extension $\hat{\phi} \in \mathrm{Irr}_+(\mathrm{Aut}(S))$.

For $S = \mathsf{A}_n$, $n \geq 5$, $n \neq 6$ ($\mathsf{A}_6 \cong \mathrm{PSL}_2(9)$) we consider the (unique) character $\phi \in \mathrm{Irr}(\mathsf{A}_n)$ such that $\phi(1) = n - 1$; then it is well known that ϕ has an extension $\hat{\phi} \in \mathrm{Irr}_+(\mathrm{Aut}(S))$.

If S is a sporadic simple group or the Tits group, we choose the unique $\phi \in \mathrm{Irr}(S)$ of degree d, where the pairs (S, d) are as follows (see [CCN+85]):

$(\mathrm{M}_{11}, 11)$, $(\mathrm{M}_{12}, 45)$, $(\mathrm{M}_{22}, 55)$, $(\mathrm{M}_{23}, 22)$, $(\mathrm{M}_{24}, 23)$, $(\mathrm{J}_1, 209)$, $(\mathrm{J}_2, 36)$, $(\mathrm{J}_3, 324)$, $(\mathrm{J}_4, 889111)$, $(\mathrm{Co}_1, 276)$, $(\mathrm{Co}_2, 23)$, $(\mathrm{Co}_3, 23)$, $(\mathrm{Fi}_{22}, 78)$, $(\mathrm{Fi}_{23}, 782)$, $(\mathrm{Fi}_{24}', 8671)$, $(\mathrm{HS}, 22)$, $(\mathrm{McL}, 22)$, $(\mathrm{He}, 680)$, $(\mathrm{Ru}, 406)$, $(\mathrm{Suz}, 143)$, $(\mathrm{O'N}, 10944)$, $(\mathrm{HN}, 760)$, $(\mathrm{Ly}, 45694)$, $(\mathrm{Th}, 248)$, $(\mathrm{B}, 4371)$, $(\mathrm{M}, 196883)$, $({}^2\mathrm{F}_4(2)', 78)$. $\square$

For a finite set X and an integer $1 \leq k \leq |X|$, we denote by

$$X^{[k]} = \{Y \subseteq X \mid |Y| = k\}$$

the set consisting of the subsets of cardinality k of X. Given an action of a group G on a set X and a subset $Y \in X^{[k]}$, we denote by $G_Y = \{g \in G \mid Y^g = Y\}$ the stabilizer of Y in the natural action of G on $X^{[k]}$.

The following result will be useful in dealing with some of the cases arising in the proof of Theorem A.

Lemma 5.2. *Let G be a finite group and let M be a non-solvable minimal normal subgroup of G such that $\mathbf{C}_G(M) = 1$. Writing $M = S_1 \times S_2 \times \cdots \times S_n$, with S_i isomorphic non-abelian simple groups, we consider the action of G on $X = \{1, 2, \ldots, n\}$ defined, for $g \in G$ and $i \in X$, by*

$$S_{i^g} = (S_i)^g \ .$$

If G satisfies Hypothesis C, then

(a) *For every non-empty $Y \subseteq X$, either G_Y is a perfect group or $|G_Y : G'_Y| = 3$.*
(b) *For every $1 \leq k \leq n$, the orbits of G on $X^{[k]}$ have distinct sizes.*

Proof. Let $X = \{1, 2, \ldots, n\}$, $S = S_1$ and $A = \mathrm{Aut}(S)$. Since $\mathbf{C}_G(M) = 1$, G is isomorphic to a subgroup of $\mathrm{Aut}(M)$. Hence, we can identify G with a subgroup of the wreath product $W = A \wr \mathrm{Sym}(X)$ ([Rob93, 3.3.20]) and the action of W on X, defined by $S_{i^w} = (S_i)^w$ for $w \in W$ and $i \in X$, extends the action of G on X. As M is a minimal normal subgroup of G, G acts transitively on X and hence there exist elements $x_1 = 1, x_2, \ldots, x_n \in G$ such that $S_i = S^{x_i}$. So, the base group of W is $B = A_1 \times A_2 \times \cdots \times A_n$, where $A_i = A^{x_i}$.

By Lemma 5.1 there exists a character $\phi \in \mathrm{Irr}_+(S)$ having an extension $\hat{\phi} \in \mathrm{Irr}_+(A)$. So, $\hat{\phi}_i = \hat{\phi}^{x_i} \in \mathrm{Irr}_+(A_i)$ extends $\phi_i = \phi^{x_i} \in \mathrm{Irr}(S_i)$, for every $i \in X$.

Let Y be a non-empty subset of X. We consider the character

$$\theta = \prod_{i \in X} \alpha_i \in \mathrm{Irr}(M), \;\text{ where } \alpha_i = \begin{cases} \phi_i & \text{if } i \in Y \\ 1_{S_i} & \text{if } i \in X \smallsetminus Y \,. \end{cases}$$

Hence, the character

$$\psi = \prod_{i \in X} \beta_i \in \mathrm{Irr}(B), \;\text{ where } \beta_i = \begin{cases} \hat{\phi}_i & \text{if } i \in Y \\ 1_{A_i} & \text{if } i \in X \smallsetminus Y \end{cases}$$

is an extension of θ to B and $\psi \in \mathrm{Irr}_+(B)$.

By [Hup98, Lemma 25.5(a)], the set stabilizer W_Y coincides with the inertia subgroup $I_W(\psi)$. Hence, as each character β_i is afforded by a real representation of A_i, the argument used in the proof of [Hup98, Lemma 25.5(b)] shows that ψ has an extension $\hat{\psi} \in \mathrm{Irr}_+(W_Y)$.

Since $\theta = \psi_M$, we have $W_Y = I_W(\psi) \leq I_W(\theta)$. On the other hand, if $w \in I_W(\theta)$, then w normalizes $\ker(\theta) = \prod_{i \in X \smallsetminus Y} S_i$ and hence $w \in W_{X \smallsetminus Y} = W_Y$. Therefore, $W_Y = I_W(\theta)$ and

$$G_Y = G \cap W_Y = G \cap I_W(\theta) = I_G(\theta).$$

Let $\hat{\theta} = \hat{\psi}_{G_Y}$ be the restriction of $\hat{\psi}$ to G_Y. As $\hat{\theta}_M = \hat{\psi}_M = \theta \in \mathrm{Irr}(M)$, $\hat{\theta}$ is an irreducible character of G_Y and, since $\hat{\psi} \in \mathrm{Irr}_+(W_Y)$, we deduce that $\hat{\theta} \in \mathrm{Irr}_+(G_Y)$.

Now, if $|G_Y/G'_Y|$ is even, then there exists a linear character λ of G_Y such that $o(\lambda) = 2$. Hence, by Gallagher's theorem and [Isa76, Lemma 4.8] $\hat{\theta}$ and $\lambda\hat{\theta}$ are distinct characters in $\mathrm{Irr}_+(G_Y)$, and by Clifford correspondence $\hat{\theta}^G$ and $(\lambda\hat{\theta})^G$ are distinct characters of the same degree in $\mathrm{Irr}_+(G)$, against Hypothesis C.

On the other hand, if $|G_Y/G'_Y| \geq 5$, then Clifford correspondence yields at least five characters of the same degree in $\mathrm{Irr}(G|\hat{\theta})$, a contradiction again. Hence, $|G_Y : G'_Y| \in \{1, 3\}$ and part (a) is proved.

So far, we have shown that for every non-empty subset Y of X there exists a character $\chi_Y \in \mathrm{Irr}_+(G)$ such that

$$\chi_Y(1) = \phi(1)^{|Y|}|G : G_Y|. \tag{5.1}$$

Moreover, the irreducible constituents of the restriction of χ_Y to M have kernels of the form

$$\prod_{i \in X \smallsetminus Y^g} S_i$$

for some $g \in G$. Hence, if $Y_1, Y_2 \in X^{[k]}$ $(1 \leq k \leq n)$ are sets belonging to different G-orbits, then $\chi_{Y_1} \neq \chi_{Y_2}$.

As $|G : G_Y|$ is the size of the orbit of $Y \in X^{[k]}$ in the action of G on $X^{[k]}$, part (b) follows immediately from (5.1) and Hypothesis C. □

Theorem 5.3. *If G is a non-solvable group and G satisfies Hypothesis C, then G is one of the groups in the list $\mathcal{L}$ defined in* (1.1).

Proof. We work by induction on $|G|$.

(I) We first assume that there exists a non-solvable minimal normal subgroup M of G, and we start by showing that $\mathbf{C}_G(M) = 1$.

Suppose, working by contradiction, that $C = \mathbf{C}_G(M) \neq 1$. Let C_0 be a minimal normal subgroup of G, with $C_0 \leq C$. As G/C_0 is non-solvable and satisfies Hypothesis C, by induction G/C_0 is a group in $\mathcal{L}$. In particular, G/C_0 is almost-simple, so $C = C_0$ is minimal normal in G, G/C has socle $MC/C \cong M$ and $|G : MC| \in \{1, 3\}$.

Assuming $G = M \times C$, then $C \cong G/M$ satisfies Hypothesis C. If C is solvable, then $C \cong C_3$ and hence, as M (by direct inspection of the socles of the groups in $\mathcal{L}$ or by [BCH92, Lemma 1]) has at least two irreducible characters with the same degree, G has at least six irreducible characters with the same degree, a contradiction. Hence, both M and C are simple groups belonging to the list $\mathcal{L}$. We observe that M and C are non-isomorphic, as otherwise G has irreducible characters with the same degree and Frobenius–Schur indicator, but with different kernels, a contradiction. Every simple group in $\mathcal{L}$, except for $\mathrm{SL}_3(2)$, M_{22}, McL and Th, has three irreducible characters with the same degree and, as G cannot have six characters of the same degree, we deduce that M and C are (non-isomorphic) groups belonging to the sublist $\{\mathrm{SL}_3(2), \mathrm{M}_{22}, \mathrm{McL}, \mathrm{Th}\}$. Now, one checks that for each of the six possibilities for $G = M \times C$, $\mathrm{Irr}(G)$ contains four non-real characters with the same degree, against Hypothesis C.

Assuming instead that $|G : MC| = 3$, then either $M \cong \mathrm{PSL}_2(8)$ or $M \cong \mathrm{O}_8^+(2)$. Hence, there are three characters $\phi_1, \phi_2, \phi_3 \in \mathrm{Irr}(M)$ with the same degree, that are transitively permuted by the outer automorphism group of order 3 of M. So, for every $\theta \in \mathrm{Irr}(C)$, $I_G(\phi_i \times \theta) = MC$ and hence $(\phi_i \times \theta)^G \in \mathrm{Irr}(G)$. If $M \cong \mathrm{SL}_2(8)$, $\phi_i(1) = 7$ and ϕ_i is non-real, while if $M \cong \mathrm{O}_8^+(2)$, $\phi_i(1) = 35$ and ϕ_i is real, for $i = 1, 2, 3$. We observe that G/M satisfies Hypothesis C and that CM/M is a minimal normal subgroup of index three of G/M. If C is solvable, then by Theorem 2.5 either $G/M \cong C_7 \rtimes C_3$ and $C \cong C_7$, or $G/M \cong \mathsf{A}_4$ and $C \cong C_2 \times C_2$. If C is non-solvable, then by induction either $C \cong \mathrm{PSL}_2(8)$ or $C \cong \mathrm{O}_8^+(2)$. In any case, there are at least three characters $\theta_1, \theta_2, \theta_3 \in \mathrm{Irr}(C)$ having the same degree and either all non-real or all real. It follows that the characters $\phi_i \times \theta_j$, for $1 \leq i, j \leq 3$, are either nine non-real, or nine real, irreducible characters with the same degree of MC. As $I_G(\phi_i \times \theta_j) = MC$, recalling [IN12, Lemma 2.1] we conclude that G has either at least three non-real, or at least three real, irreducible characters with the same degree, against Hypothesis C.

So far, we have proved that $\mathbf{C}_G(M) = 1$ and hence M is the only minimal normal subgroup of G. Writing

$$M = S_1 \times \cdots \times S_n,$$

with $S_i \simeq S$ isomorphic non-abelian simple groups, we define

$$N = \bigcap_{i=1}^{n} \mathbf{N}_G(S_i), \text{ and } H = G/N.$$

So, H is a transitive permutation group on $X = \{1, \dots, n\}$. If $n = 1$, then G is an almost simple group and $G \in \mathcal{L}$ by Theorem 3.1. We now assume, aiming at a contradiction, that $n \neq 1$. We observe that then, in particular, $H \neq 1$. We have two cases.

H	$C_7 \rtimes C_3$	$C_2^3 \rtimes (C_7 \rtimes C_3)$	A_4	$\mathrm{SL}_2(3)$	$C_3^2 \rtimes \mathrm{SL}_2(3)$	$C_5^2 \rtimes \mathrm{SL}_2(3)$
$N(H)$	$\{7\}$	$\{8, 14, 56\}$	$\{4\}$	$\{8\}$	$\{9, 72\}$	$\{25, 200\}$
$W(H)$	$\{7, 21\}$	$\{8, 14, 56, 168\}$	$\{4, 12\}$	$\{8, 24\}$	$\{9, 72, 216\}$	$\{25, 200, 600\}$

TABLE 3.

(I.a) We first assume that G/M is solvable. Hence, for every non-empty subset Y of X, by part (a) of Lemma 5.2 either $G_Y = M$ or $[G_Y : G'_Y] = 3$. By Theorem 2.5, both G/M and $H = G/N$ belong to the list

$$\mathcal{S} = \begin{array}{l}\big\{C_3, C_7 \rtimes C_3, (C_2 \times C_2 \times C_2) \rtimes (C_7 \rtimes C_3), \mathsf{A}_4, \\ \mathrm{SL}_2(3), (C_3 \times C_3) \rtimes \mathrm{SL}_2(3), (C_5 \times C_5) \rtimes \mathrm{SL}_2(3)\big\}.\end{array}$$

We remark that, as H_Y is a quotient of G_Y,

$$\text{for all } \; Y \subseteq X, Y \neq \varnothing, \; \text{ either } \; H_Y = 1 \; \text{ or } \; |H_Y : H'_Y| = 3\,. \tag{5.2}$$

Let $L = H_x$, for some $x \in X$, be a point stabilizer in H; so $n = |X| = |H : L|$. We first show that H does not act regularly on X, i.e. that $L \neq 1$. To prove this, we assume that H acts regularly on X and we consider a subset $Y = \{x_1, x_2\} \in X^{[2]}$. For $h \in H_Y$, then h^2 fixes x_i, for $i = 1, 2$, so $h^2 = 1$. It follows that H_Y is an elementary abelian group, and hence $H_Y = 1$ by (5.2). Thus, H has $(n-1)/2$ orbits of size n on $X^{[2]}$ and part (b) of Lemma 5.2 yields $n = 3$ and $H \cong C_3$. Moreover, as $G/M \in \mathcal{S}$ and $G_Y/M = N/M$, part (a) of Lemma 5.2 implies that $N = M$. In particular, for every $\alpha \in \mathrm{Irr}(S)$, $\alpha \times \alpha \times \alpha \in \mathrm{Irr}(M)$ has three extensions to $\mathrm{Irr}(G)$. Since S has at least two irreducible characters with the same degree by [BCH92, Lemma 1], then G has at least six irreducible characters with the same degree, a contradiction.

Hence, $L \neq 1$ and, as H acts faithfully on X, the normal core L_H of L in H is trivial. Recalling that $L < H$ since $n \neq 1$ and that $|L/L'| = 3$ by (5.2), we conclude that

$$|X| \in N(H) = \big\{|H : L| \mid 1 \neq L < H, |L/L'| = 3 \text{ and } L_H = 1\big\}.$$

In order to reduce the number of cases for the relevant transitive actions of the groups in $\mathcal{S}$, we introduce the set

$$W(H) = \big\{|H : T| \mid T < H \text{ and } |T/T'| = 3\} \cup \{|H|\big\}$$

and we claim that, for every $0 < k < n$, the size $|H : H_Y|$ of the H-orbit of $Y \in X^{[k]}$ belongs to $W(H)$. In fact, $H_Y < H$ as H is transitive on X and $Y \neq X$, so the claim follows by (5.2).

Thus, recalling that $n = |X| \in N(H)$, part (b) of Lemma 5.2 yields that, for every k such that $0 < k < n$,

$$|X^{[k]}| = \binom{n}{k} \text{ is a sum of distinct elements of } W(H). \tag{5.3}$$

Since $H \not\cong C_3$ as H acts non-regularly on X, the triples $(H, N(H), W(H))$ are listed in Table 3.

One easily checks that (5.3) is not satisfied for $k = 3$ if $H \cong C_7 \rtimes C_3$, while it is not satisfied for $k = 2$ in all other cases, a contradiction.

(I.b) We assume now that G/M is non-solvable. Since G/M satisfies Hypothesis C, by induction $G/M \in \mathcal{L}$. We observe that N/M, being isomorphic to a subgroup of a direct

H	A_8	$\mathrm{PSL}_3(2)$	M_{11}	M_{22}	M_{23}	M_{24}	$\mathrm{PSL}_2(8).3$
$b(\|H\|)$	18	8	16	22	28	32	14
$N_1(H)$	$\{8,15\}$	$\{7,8\}$	$\{11,12\}$	$\{22\}$	$\{23\}$	$\{24\}$	$\{9\}$

TABLE 4.

product of outer automorphism groups of a simple group, is a solvable normal subgroup of G/M. Hence, $N = M$ and $H = G/M$.

In order to reduce the number of cases for the action of H on X, we consider the function

$$b(|H|) = \max\Big\{a \in d(|H|) \mid \sum_{1 \neq d \in d(|H|)} d \geq \binom{a}{\lfloor \frac{a}{2} \rfloor}\Big\}$$

where $d(|H|)$ is the set of the positive divisors of $|H|$. By the orbit formula and part (b) of Lemma 5.2, setting $m = \lfloor \frac{|X|}{2} \rfloor$, $|X^{[m]}|$ is a sum of distinct elements (orbit sizes) $d_i \in d(|H|)$. Moreover, every $d_i \neq 1$, because H has no fixed points in $X^{[m]}$ since H is transitive on X and $0 < m < |X|$.

We deduce that $|X| \leq b(|H|)$ and hence $|X|$ belongs to the set

$$N_1(H) = \big\{c \cdot |H:L| \mid c \in \mathbb{Z}_+, L \text{ is a maximal subgroup of } H \text{ and } c \cdot |H:L| \leq b(|H|)\big\}.$$

We remark that, for $H \in \mathcal{L}$, both $b(|H|)$ and $N_1(H)$ can be computed using the information in [CCN+85] and that the set stabilizers H_Y for the relevant actions of H on $X^{[k]}$, for $|X| \in N_1(H)$ and $0 < k < |X|$, can be computed in [GAP21]. In particular, we get that $N_1(H) = \varnothing$ for

$$H \in \big\{\mathrm{SU}_3(3), \mathrm{McL}, \mathrm{Th}, \mathrm{O}_8^+(2).3\big\}.$$

For the remaining cases, we refer to Table 4.

For $(H, |X|) \in \{(\mathrm{PSL}_3(2), 7), (\mathrm{M}_{11}, 11), (\mathrm{M}_{11}, 12)\}$ and $Y = \{x\} \subseteq X$, H_Y has a quotient of order 2, a contradiction by part (a) of Lemma 5.2. In all other cases, for $Y \in X^{[2]}$, H_Y has a quotient of order 2, again a contradiction.

(**II.**) We can now assume that G has no non-solvable minimal normal subgroups. So, M is solvable and G/M is a non-solvable group that satisfies Hypothesis C. Hence, by induction G/M is an almost-simple group belonging to the list $\mathcal{L}$. Let K/M be the socle of G/M.

We observe that $M \leq K'$. In fact, $K'M/M = (K/M)' = K/M$, hence $K = K'M$, and $K' \trianglelefteq G$, so if $M \not\leq K'$, then $M \cap K' = 1$ and $K' \cong K/M$ is a non-solvable minimal normal subgroup of G, a contradiction.

If $M < \mathbf{C}_G(M)$, then $M \leq \mathbf{Z}(K)$ and hence K is a quotient of the Schur cover of K/M. If $G = K$, i.e. G/M is a simple group in $\mathcal{L}$, by direct inspection using [CCN+85], we see that all the groups arising in this way do not satisfy Hypothesis C. Since the Schur multiplier of $\mathrm{SL}_2(8)$ is trivial, the only case left is $G/M \cong \mathrm{O}_8^+(2).3$. Hence, $|M| = 2^2$ (as the non-trivial elements of the Schur multiplier of $\mathrm{O}_8^+(2)$ are transitively permuted by the subgroup of order three of its outer automorphism group) and G has three real irreducible characters of degree 3024, a contradiction.

Therefore, M is the unique minimal normal subgroup of G and it is a faithful non-trivial irreducible G/M-module. Hence, an application of Theorem 4.1 gives the final contradiction, concluding the proof. □

Now Theorem A follows immediately from Theorem 2.5 and Theorem 5.3.

References

[BCH92] Yakov Berkovich, David Chillag, and Marcel Herzog, *Finite groups in which the degrees of the nonlinear irreducible characters are distinct*, Proc. Amer. Math. Soc. **115** (1992), no. 4, 955–959. MR 1088438

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR 1484478

[BEOH23] Hans Ulrich Besche, Bettina Eick, Eamonn O'Brien, and Max Horn, *SmallGrp, the GAP small groups library, Version 1.5.3*, `https://gap-packages.github.io/smallgrp/`, May 2023, GAP package.

[Bre24] Thomas Breuer, *CTblLib, the GAP character table library, Version 1.3.9*, `https://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib`, Mar 2024, GAP package.

[CCN+85] John H. Conway, Robert T. Curtis, Simon P. Norton, Richard A. Parker, and Robert A. Wilson, $\mathbb{ATLAS}$ *of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray. MR 827219

[Fei93] Walter Feit, *Extending Steinberg characters*, Linear algebraic groups and their representations (Los Angeles, CA, 1992), Contemp. Math., vol. 153, Amer. Math. Soc., Providence, RI, 1993, pp. 1–9. MR 1247494

[FMOW19] Joanna B. Fawcett, Jürgen Müller, Eamonn A. O'Brien, and Robert A. Wilson, *Regular orbits of sporadic simple groups*, J. Algebra **522** (2019), 61–79. MR 3893366

[FOS16] Joanna B. Fawcett, Eamonn A. O'Brien, and Jan Saxl, *Regular orbits of symmetric and alternating groups*, J. Algebra **458** (2016), 21–52. MR 3500766

[FS89] Walter Feit and Gary M. Seitz, *On finite rational groups and related topics*, Illinois J. Math. **33** (1989), no. 1, 103–131. MR 974014

[GAP21] *GAP – groups, algorithms, and programming, version 4.11.1*, `https://www.gap-system.org`, 2021.

[Hol23] Derek Holt, *cohomolo, cohomology groups of finite groups on finite modules, Version 1.6.11*, `https://gap-packages.github.io/cohomolo`, Jan 2023, GAP package.

[Hul01] Alexander Hulpke, *Representing subgroups of finitely presented groups by quotient subgroups*, Experimental Mathematics **10** (2001), no. 3, 369–381.

[Hup67] Bertram Huppert, *Endliche Gruppen. I*, Die Grundlehren der mathematischen Wissenschaften, vol. Band 134, Springer-Verlag, Berlin-New York, 1967. MR 224703

[Hup98] ———, *Character theory of finite groups*, De Gruyter Expositions in Mathematics, vol. 25, Walter de Gruyter & Co., Berlin, 1998. MR 1645304

[IN12] I. Martin Isaacs and Gabriel Navarro, *Groups whose real irreducible characters have degrees coprime to p*, J. Algebra **356** (2012), 195–206. MR 2891128

[Isa76] I. Martin Isaacs, *Character theory of finite groups*, Pure and Applied Mathematics, No. 69, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976. MR 460423

[Jan05] Christoph Jansen, *The minimal degrees of faithful representations of the sporadic simple groups and their covering groups*, LMS J. Comput. Math. **8** (2005), 122–144. MR 2153793

[JK81] Gordon James and Adalbert Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, MA, 1981, With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson. MR 644144

[JLPW95] Christoph Jansen, Klaus Lux, Richard Parker, and Robert Wilson, *An atlas of Brauer characters*, London Mathematical Society Monographs. New Series, vol. 11, The Clarendon Press, Oxford University Press, New York, 1995, Appendix 2 by T. Breuer and S. Norton, Oxford Science Publications. MR 1367961

[MNP24] Thomas Merkwitz, Liam Naughton, and Götz Pfeiffer, *TomLib, the GAP library of tables of marks, Version 1.2.11*, `https://gap-packages.github.io/tomlib`, Jan 2024, GAP package.

[OSC24] *Oscar – open source computer algebra research system, version 1.2.0*, `https://www.oscar-system.org`, 2024.

[Rob93] Derek J. S. Robinson, *A course in the theory of groups*, Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1993. MR 1261639

[Ser77] Jean-Pierre Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, Vol. 42, Springer-Verlag, New York-Heidelberg, 1977, Translated from the second French edition by Leonard L. Scott. MR 450380

[Tre17] Stephen Trefethen, *Non-abelian composition factors of m-rational groups*, J. Algebra **485** (2017), 288–309. MR 3659336

[Wol78] Thomas R. Wolf, *Character correspondences in solvable groups*, Illinois J. Math. **22** (1978), no. 2, 327–340. MR 498821

[WWT+] Robert Wilson, Peter Walsh, Jonathan Tripp, Ibrahim Suleiman, Richard A. Parker, Simon P. Norton, Simon Nickerson, Steve Linton, John Bray, and Rachel Abbott, *Atlas of finite group representations*, `http://atlas.math.rwth-aachen.de/Atlas/v3`.

T. Breuer: Lehrstuhl für Algebra und Zahlentheorie, RWTH Aachen Universty, Germany
Email address: `sam@math.rwth-aachen.de`

F. Calegari: Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
Email address: `fcale@uchicago.edu`

S. Dolfi: Dipartimento di Matematica, Università di Firenze, 50134 Firenze, Italy
Email address: `silvio.dolfi@unifi.it`

G. Navarro: Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain
Email address: `gabriel@uv.es`

P. H. Tiep: Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
Email address: `tiep@math.rutgers.edu`