Counting Perron numbers by absolute value

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Abstract

We count various classes of algebraic integers of fixed degree by their largest absolute value. The classes of integers considered include all algebraic integers, Perron numbers, totally real integers, and totally complex integers. We give qualitative and quantitative results concerning the distribution of Perron numbers, answering in part a question of Thurston ['Entropy in dimension one', to appear in the *Proceedings of the Banff Conference on Frontiers in Complex Dynamics*].

1. *Introduction*

The goal of this paper is to address several questions concerning Perron numbers suggested by a recent preprint of Thurston [**12**]. An algebraic integer α is a *Perron number* if it has larger absolute value than any of its Galois conjugates. How many Perron numbers are there? Although there are numerous ways to order and count algebraic integers, in this context it seems most natural to count (in fixed degree) by absolute value, and this is what we do. As well as counting Perron algebraic integers, we count all algebraic integers. Let $\mathcal{A}_N$ denote the set of algebraic integers of degree N. For $\alpha \in \mathcal{A}_N$, let $\overline{|\alpha|}$ — the *house* of α — denote the largest absolute value of any conjugate $\sigma\alpha$ of α. An argument of Kronecker shows that there are only finitely many elements of $\mathcal{A}_N$ with house at most X. Specifically, any algebraic integer of degree N all of whose conjugates' absolute values is bounded by X is the root of a polynomial whose coefficients are bounded strictly in terms of X and N, and hence there are only finitely many such α. Let $\mathcal{A}_N^+ \subset \mathcal{A}_N$ denote the subset of totally real algebraic integers of degree N. Let $\mathcal{A}_N^- \subset \mathcal{A}_N$ denote the subset of algebraic integers of degree N which are totally complex (this is empty unless N is even). Let $\mathcal{A}_N^P \subset \mathcal{A}_N$ denote the algebraic integers of degree N which are Galois conjugate to a Perron number. Our first result gives an estimate for the sizes of these sets.

THEOREM 1.1. *As $X \to \infty$,*

$$|\mathcal{A}_N^*(X)| = X^{N(N+1)/2} D_N^* \left(1 + O\left(\frac{1}{X}\right)\right),$$

where $*$ is either unadorned, P, $+$, or $-$ when N is even, and the constants D_N^* are given as follows:

$$D_N = \prod_{k=0}^{m-1} \left(\frac{k!^2 2^{2k+1}}{(2k+1)!}\right)^2, N = 2m; \quad D_N = \left(\frac{m!^2 2^{2m+1}}{(2m+1)!}\right) \prod_{k=0}^{m-1} \left(\frac{k!^2 2^{2k+1}}{(2k+1)!}\right)^2, N = 2m+1,$$

$$D_N^P = \frac{D_N}{N+1}, N = 2m; \quad D_N^P = \frac{D_N}{N}, N = 2m+1,$$

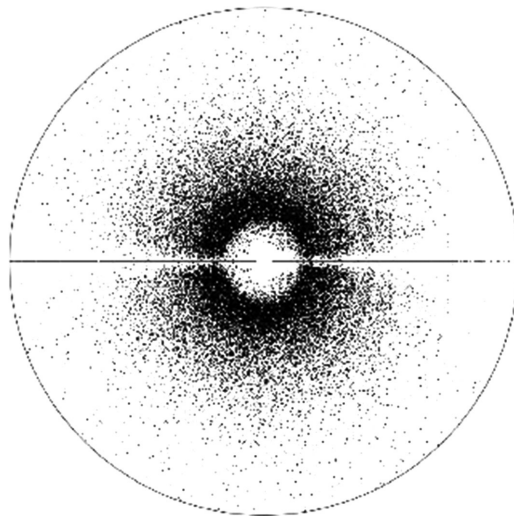$$D_N^+ = \prod_{k=0}^{N-1} \frac{2^{k+1}k!^2}{(2k+1)!}, \quad D_{2N}^- = \frac{2^{2N(N-1)}(2N)!}{N!^2} D_{2N}^+.$$

FIGURE 1. *A plot from* [12] *showing the normalized roots* $\sigma\alpha/\alpha$ *of the minimal polynomials for 5932 degree* 21 *Perron numbers* α, *obtained by sampling* 20 000 *monic degree* 21 *polynomials with integer coefficients in* $[-5, 5]$ *and keeping those that have a root of absolute value at most* 5 *which is larger than all other roots.*

For example, given a 'random' algebraic integer, the probability that α is (conjugate to) a Perron algebraic integer is $1/N$ if N is odd and $1/(N+1)$ if N is even. Note that the last answer with the ratio D_{2N}^-/D_{2N}^+ proves Conjecture 5.2 in [1]. This theorem reduces to understanding various integrals over the region Ω_N in $\mathbf{R}^N$ which parameterize monic polynomials all of whose roots have absolute value at most 1. If one imposes conditions on the signature, then one obtains corresponding regions $\Omega_{R,S} \subset \Omega_N$ with $N = R + 2S$, the constants D_N^+ and D_{2N}^- are then the volumes of $\Omega_{N,0}$ and $\Omega_{0,N}$, respectively. There is a natural decomposition

$$\Omega_N = \coprod_{R+2S=N} \Omega_{R,S}.$$

Beyond counting algebraic integers in these classes, it is also of interest to try to understand what a 'typical' such element is, under the constraint that the house of α is bounded by a fixed constant X, which leads us toward our next result.

1.1. A question of Thurston

Thurston asked [12] whether one could understand the distribution of Perron numbers subject to the constraint that their absolute values are bounded by a fixed real number X. Recall that a Perron algebraic integer is a real algebraic integer α with $|\alpha| = \overline{|\alpha|}$ whose absolute is strictly larger than all its Galois conjugates. We say that a polynomial with coefficients in $\mathbf{R}$ is Perron if it has a unique (necessarily real) largest (in terms of absolute value) root. Usually one insists that a Perron number is a *positive* real number, but with our definition α is Perron if and only if $-\alpha$ is also Perron. (The only change to the asymptotics is a factor of 2.) We explain in Section 1.8 how to count Perron algebraic integers. In one experiment, Thurston attempted to model random polynomials whose largest root is $\leqslant 5$ by taking polynomials of degree 21 all of whose coefficients lie in the interval $[-5, 5]$. The corresponding roots showed a tendency to cluster the ratio of their absolute values to the largest root away from $|z| = 1$; we include his figure here as Figure 1. However, we shall explain why this picture is not an accurate representation of the entire space of Perron polynomials. As a point of comparison, Thurston sampled polynomials over a space with 11^{21} lattice points and volume 10^{21}. On the
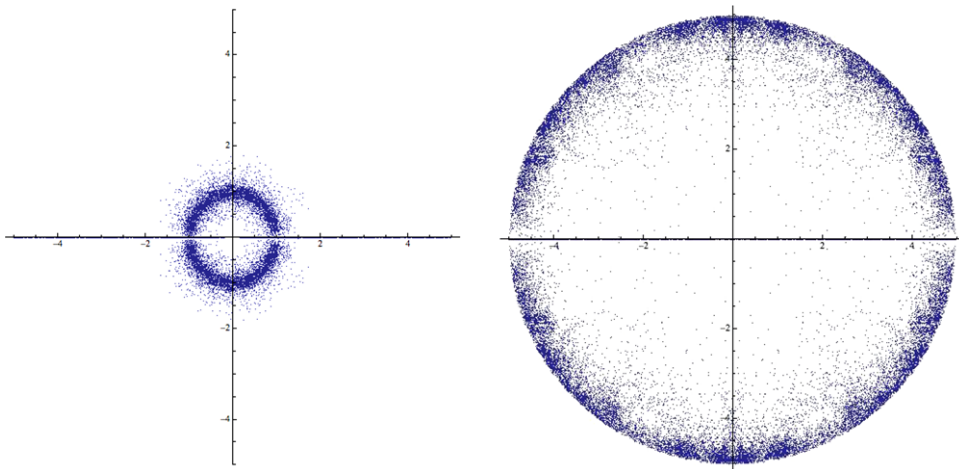
FIGURE 2 (colour online). *The first graph consists of the roots of Perron polynomials with largest root $\leqslant 5$ and integral coefficients in $[-5, 5]$ as in [12], except no longer normalizing by the largest (necessarily real) root α. The second graph consists of roots of random Perron polynomials in Ω_{21}^P scaled to have a maximal root of absolute value $\leqslant 5$ as generated by a random walk Metropolis–Hastings algorithm. The second graph is in accordance with Theorem 1.2.*

other hand, let $\Omega_{21}^P \subset \Omega_{21}$ be the region consisting of polynomials with a unique largest root, and consider the scaled version of this space where the roots are allowed to have absolute value at most 5. Then this region has volume

$$5^{210} \cdot D_{21}^P = \frac{2^{189} 5^{198}}{3^{24} 7^{10} 11^{11} 13^9 17^5 19^3} \sim 8.308 \times 10^{143}.$$

Hence Thurston's samples are taken from a region which represents less than one 10^{123}th of the entire space of Perron polynomials. As another illustration, the average value of $|a_{21}|$ over the correct region is

$$5^{21} \cdot \frac{88179}{524288} = \frac{3 \cdot 5^{21} \cdot 7 \cdot 13 \cdot 17 \cdot 19}{2^{19}} \sim 8.020 \times 10^{13},$$

which is not anywhere close to being in $[-5, 5]$. Indeed, this value might *a priori* be considered surprisingly large, given that the absolute maximum of the constant term $|a_{21}|$ is $5^{21} \sim 4.768 \times 10^{14}$. We should make clear that Thurston made no claims that his experiment produced a faithful representation of Ω_{21}^P, and he explicitly mentions the coefficients of a typical member of Ω_{21}^P appears to be 'much larger' than 5. Indeed, one of the problems he posed is to formulate a good method for sampling 'randomly' in this space. A natural approach to the latter question is to use a random walk Metropolis–Hastings algorithm. Figure 2, produced via such a random walk algorithm, is in agreement with our theoretical results, such as Theorem 1.2 below. The 'ring' structure evident in Thurston's picture (of radius approximately 1/5) is a consequence of the fact that polynomials with (suitably) small coefficients have roots which tend to cluster uniformly around the disc of radius one. This follows in the radial direction by a theorem of Erdös and Turán [6], and for the absolute values from [8]. In contrast, the reality is that the conjugates of Perron polynomials will cluster around the boundary, which is our second result:

THEOREM 1.2. *As $N \to \infty$, the roots of a random polynomial in Ω_N or Ω_N^P are distributed uniformly about the unit circle.*

1.2. *Asymptotics*

It is easy to give asymptotics for any product formula using Stirling's formula and its variants. For example, we have the following:

LEMMA 1.3. *The probability that a polynomial $P(x) \in \Omega_N$ has only real roots is*

$$\frac{D_N^+}{D_N} \sim \frac{C \cdot N^{1/8}}{2^{N^2/2}}.$$

The probability that a polynomial $P(x) \in \Omega_{2N}$ has no real roots, equivalently, that $P(x) > 0$ for all x, is

$$\frac{D_{2N}^-}{D_{2N}} \sim \frac{2C}{\sqrt{2\pi} \cdot (2N)^{3/8}}$$

for the same constant C as above.

REMARK 1.4. With a little extra care, one can also identify the constant C above as

$$C = 2^{-1/24} e^{-3/2 \cdot \zeta'(-1)} = 1.24514\ldots.$$

REMARK 1.5. A polynomial $P(x) \in \Omega_{2N}$ is positive everywhere if and only if it is positive on $[-1, 1]$. On the other hand, for many classes of random models of polynomials, it is a theorem of Dembo, Poonen, Shao, and Zeitouni [5] that a random polynomial whose coefficients are chosen with (say) identical normal distributions with zero mean is positive in $[-\infty, \infty]$ with probability $N^{-b+o(1)}$ and positive in $[-1, 1]$ with probability $N^{-b/2+o(1)}$ for some universal constant $b/2$, which they estimate be 0.38 ± 0.015. On the other hand, the exponent occurring above is $3/8 = 0.375$. Is there any direct relationship between these theorems? For example, does this suggest that $b = 3/4$?

1.3. *The limit $N \to \infty$*

As $N \to \infty$, we are still able to say something about the geometric spaces Ω_N, but the direct connection with algebraic integers becomes more tenuous. Given a fixed region Ω with appropriate properties, it is quite reasonable to be able to count lattice points in the large X limit as Ω is scaled appropriately. However, the error in any such estimate will depend highly on Ω, so this does not allow one to understand the lattice points in a sequence Ω_N of regions simply in terms of the volume. There are some known global constraints. For example, Kronecker proved that the only elements of $\mathcal{A}_N$ with house in $[0, 1]$ are roots of unity, and the only elements of $\mathcal{A}_N^+$ with house in $[-2, 2]$ are of the form $\zeta + \zeta^{-1}$ for a root of unity ζ. This is consistent with our volume computations; the smallest value of X for which $\text{Vol}(\Omega_N)X^{N(N+1)/2}$ is $\geqslant 1$ is

$$1 + \frac{\log N}{N} + O\left(\frac{1}{N}\right),$$

whereas the corresponding value for $\text{Vol}(\Omega_{N,0})X^{N(N+1)/2}$ is

$$2 + \frac{2\log N}{N} + O\left(\frac{1}{N}\right).$$

(In practice, there exist algebraic integers which are not roots of unity of house at least as small as $2^{1/N} \sim 1 + \log 2/N$.)

1.4. Configuration spaces

A natural way to understand the spaces $\Omega_{R,S}$ is to consider the spaces defined by the roots. In this way, one can relate integrals over $\Omega_{R,S}$ to integrals over nicer spaces at the expense of including the factor coming from the Jacobian. For example, consider the case of $\Omega_{N,0}$. There is a natural map:

$$[-1,1]^N \to \mathbf{R}^N,$$

given by

$$\phi : (x_1, \ldots, x_N) \to (s_1, \ldots, s_N),$$

where s_m is the mth symmetric polynomial in the x_is. Suppose that

$$s_{m,k} := \frac{\partial}{\partial x_k} s_m.$$

Then $s_{m,k}$ is the $(m-1)$th symmetric polynomial in the variables x_i with x_k omitted, and the Jacobian matrix is given by $J(\phi_*) = [s_{m,k}]$. If $x_i = x_j$ then $s_{m,i} = s_{m,j}$ and the Jacobian vanishes. By comparing degrees, it follows that $|J(\phi_*)|$ is the absolute value of the Vandermonde determinant. Since ϕ is generically $N!$ to 1, it follows that

$$\int_{\Omega_{N,0}} dV = \frac{1}{N!} \int_{[-1,1]^N} \prod |x_i - x_j| dx_1 \ldots dx_N = D_N^+.$$

Yet the latter integral can be computed exactly because it is a special case of the integrals considered by [**10**]. This is enough to prove the corresponding claim in Theorem 1.1 in this case. Similar parameterizations allow us to write $\int_{\Omega_{R,S}}$ as a multiple integral, but not all the integrals which arise have such nice product expressions.

1.5. Selberg integrals

We now consider all monic polynomials with real coefficients whose roots have absolute value at most one. We assume that the polynomial has degree N, and that the polynomial has R real roots and S pairs of complex roots. Let $B(1)$ be the unit ball in $\mathbf{C}$. There is a map

$$B(1)^S \times [-1,1]^R \to \Omega_{R,S} \subset \mathbf{R}^N,$$

given by

$$\phi : (z_1, \ldots, z_S, x_1, \ldots, x_R) \to (z_1, \ldots, z_S, \overline{z}_1, \ldots, \overline{z}_S, x_1, \ldots, x_R) \to (s_1, \ldots, s_N),$$

where the s_i are symmetric in the N variables. The following is elementary:

LEMMA 1.6. *The absolute value of the determinant $|J(\phi_*)|$ is the absolute value of*

$$2^S \prod_{i=1}^S (z_i - \overline{z_i}) \prod_{i \neq j} (z_i - z_j)(z_i - \overline{z_j}) \prod_{i \neq j} (x_i - x_j) \prod_{i,j} (x_i - z_j)(x_i - \overline{z_j}).$$

The mapping from $B(1)^S \times [-1,1]^R$ to $\Omega_{R,S}$ is not 1 to 1. Rather, there is a generically faithful transitive action of the group $\mathbf{Z}/2\mathbf{Z} \wr S_s \times S_r$ on the fibers. Hence

$$\int_{\Omega_{R,S}} dV := \frac{1}{R!S!} \int_{[-1,1]^R} \int_{B(1)^S} \prod_{i=1}^S |z_i - \overline{z_i}| \prod_{i>j} |(z_i - z_j)(z_i - \overline{z_j})|^2 \prod_{i>j} |x_i - x_j| \prod_{i,j} |x - z_j|^2 dV.$$

We shall discuss various integrals which are analogues of certain Selberg integrals. These correspond to the contexts in which the roots are totally real, totally imaginary, or without any restriction.

DEFINITION 1.7. Let $C_N(\alpha, T)$ denote the integral

$$\int_{\Omega_N} |a_N|^{\alpha-1} P(T) dV.$$

THEOREM 1.8. *There are equalities as follows. If $N = 2m + 1$, then*

$$C_N(\alpha, T) = D_N \left(\prod_{k=0}^{m} \frac{1+2k}{\alpha+2k} \right) \cdot \sum_{i=0}^{m} \frac{2i+1}{2m+1} \frac{m!^2}{(2m)!} \binom{2i}{i} \binom{2m-2i}{m-i} T^{2i+1}.$$

If $N = 2m$, then

$$C_N(\alpha, T) = D_N \cdot \left(\prod_{k=0}^{m-1} \frac{1+2k}{\alpha+2k} \right) \cdot \sum_{i=0}^{m} \frac{2i+\alpha}{2m+\alpha} \frac{m!^2}{(2m)!} \binom{2i}{i} \binom{2m-2i}{m-i} T^{2i}.$$

This theorem is proved in § 3. By considering the leading term of the polynomial, this integral formula implies that

$$\int_{\Omega_N} |a_N|^{\alpha-1} dV = D_N \left(\prod_{k=0}^{m} \frac{1+2k}{\alpha+2k} \right) = D_N \frac{(1/2)_{m+1}}{(\alpha/2)_{m+1}},$$

where $N = 2m$ or $2m + 1$. Specializing further to $\alpha = 1$, we deduce that $\int_{\Omega_N} dV = D_N$. This latter integral was also computed by Fam [7], the evaluation of $C_N(\alpha, T)$ above is similar.

1.6. *Comparison with classical Selberg integrals*

In § 1.9, we define integrals C_N^+ and C_N^- which are similar to $C_N(\alpha, T)$ except the integral takes place over $\Omega_{N,0}$ or $\Omega_{0,N}$ respectively. The integral $C_N(\alpha, T)$ is in some sense both the most natural (in that they are integrals over the entire space Ω_N) and unnatural, in that they are most naturally written as a sum of multiple integrals, not each of which obviously admits an exact formula. The real Selberg integrals are closest to the classical Selberg integrals, but even in this case they are not obvious specializations of known integrals. To explain this further, recall that Selberg's integral is a generalization of the β-integral, which we write as

$$\int_{-1}^{1} (1+t)^{\alpha-1} (1-t)^{\beta-1} dt = \frac{2^{\alpha+\beta-1} \Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Selberg's integral $\int_{[0,1]^N} \prod t_i^{\alpha-1} (1-t_i)^{\beta-1} \prod |t_i - t_j|^{2\gamma} dt_1 \ldots dt_N$ can be written (up to easy factors) as

$$\int_{\Omega_{N,0}} P(1)^{\alpha-1} P(-1)^{\beta-1} |\Delta_P|^{2\gamma-1},$$

where we integrate over the configuration space of monic polynomials P of degree N with real roots in $[-1, 1]$, and Δ_P is the discriminant of the corresponding polynomial. On the other hand, when one writes down similar integrals over Ω_N, the corresponding integrals do not have nice product expressions. To take a simple example, one finds that

$$\frac{1}{3!} \int_{[-1,1]^3} \prod (x_i - x_j)^2 dx_1\, dx_2\, dx_3 + \int_0^{2\pi} \int_0^1 \int_{-1}^1 4r^3 \sin^2(\theta)(x^2 - 2xr\cos(\theta) + r^2)^2\, dx\, dr\, d\theta$$

$$= \int_{\Omega_3} |\Delta_P| = \frac{32}{135} + \frac{41\pi}{15}.$$

The range of suitable integrals which have nice expressions over Ω_N seems to be fairly limited. Curiously enough, however, the integrals we do define do not even specialize to the β-integral over $\Omega_{1,0} = [-1, 1]$, but rather to

$$\int_{-1}^{1} |t|^{\alpha-1} dt = \frac{2}{\alpha},$$

and mild variants thereof. Moreover, there does not seem to be any flexibility in varying the power of the discriminant, as evidenced by the example over Ω_3 above. On the other hand, one does have the following integral over Ω_N, (Theorem 3.1), which is the direct generalization of Selberg's integral when $\gamma = 1/2$:

$$S_N(\alpha, \beta) = \int_{\Omega_N} P(1)^{\alpha-1} |P(-1)|^{\beta-1}$$

$$= \prod_{k=1}^{2m} \frac{2^{\alpha+\beta+k-2}}{\Gamma(\alpha+\beta+k-1)} \prod_{k=1}^{m} \Gamma(\alpha+\beta+k-1)\Gamma(k)$$

$$\times \prod_{k=1}^{m} \Gamma(\alpha+k+1)\Gamma(\beta+k-1), \ N = 2m \text{ even},$$

$$\prod_{k=1}^{2m+1} \frac{2^{\alpha+\beta+k-2}}{\Gamma(\alpha+\beta+k-1)} \prod_{k=1}^{m} \Gamma(\alpha+\beta+k-1)\Gamma(k)$$

$$\times \prod_{k=1}^{m+1} \Gamma(\alpha+k+1)\Gamma(\beta+k-1), \ N = 2m+1 \text{ odd}.$$

1.7. Moments

The integral $C_N(\alpha) = \int_{\Omega_N} |a_N|^{\alpha-1} dV$ allows a precise description of the moments of $|P(0)| = |a_N|$ on Ω_N, namely

$$E(\Omega_N, |a_N|^{\alpha-1}) = \frac{\int_{\Omega_N} |a_N|^{\alpha-1} dV}{\int_{\Omega_N} dV} = \prod_{k=0}^{\lfloor (N-1)/2 \rfloor} \frac{1+2k}{\alpha+2k}.$$

Denote this function by $M_N(\alpha)$. Suppose that $\mu_N = H_N(x)dx$ is the distribution on $[0, 1]$ of $|a_N|$ over Ω_N. Then we know

$$M_N(\alpha) := \int_0^1 x^{\alpha-1} H_N(x) dx = \int_0^\infty e^{-\alpha t} H_N(e^{-t}) dt.$$

It follows that $H_N(e^{-t})$ is the inverse Laplace transform of $M(\alpha)$. Write $m = \lfloor (N-1)/2 \rfloor$.

LEMMA 1.9. *The measure $\mu_N = H_N(x)dx$ is given on $[0, 1]$ by*

$$\frac{(1-x^2)^m}{\left(\int_0^1 (1-x^2)^m dx \right)} dx = \frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma(m+3/2)}{\Gamma(m+1)} (1-x^2)^m dx.$$

By taking the logarithmic derivative with respect to α and then sending α to 1, we also find that, with the same m,

$$E(\Omega_N, \log|a_N|) = -\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2m+1}\right) = -\frac{1}{2}\left(\log 2N + \gamma\right) + O\left(\frac{1}{N}\right),$$

and hence, for any root β,

$$E(\Omega_N, \log|\beta|) = -\frac{\log N}{2N} + O\left(\frac{1}{N}\right).$$

Since $|x| - 1 \geqslant \log|x|$ for $x \in B(1)$, we obtain the estimate:

$$E(\Omega_N, |\beta|) \geqslant 1 - \frac{\log N}{2N}.$$

It follows that the expected number of roots with absolute value less than $1 - \epsilon/2$ is less than $\log|N|\epsilon$; in particular, it follows that almost all roots have absolute value within $1/N^\delta$ of 1 for any fixed $\delta < 1$ as $N \to \infty$.

1.8. Perron algebraic integers

If α is a Perron algebraic integer, then necessarily α is real, since otherwise $|\alpha| = |\sigma\alpha|$. The converse is not true, however (take $\alpha = 2^{1/4}$). The natural model for Perron numbers is polynomials whose largest root is real. (The subspace of polynomials with more than one non-conjugate largest root has zero measure.) Let Ω_N^P and $\Omega_{R,S}^P$ denote the corresponding spaces. Then there is a natural map:

$$\Omega_{N-1} \times [-1,1] \to \Omega_N^P$$

given by sending $P(x)$ to $t^{N-1}P(x/t)(x-t)$. The effect on the variables is

$$b_k = t^k(a_k - a_{k-1}).$$

The Jacobian of this matrix is $(-1)^N \cdot t^{(N-1)(N+2)/2}P(1)$, and so, if $N = R + 2S$,

$$\int_{\Omega_{R,S}^P} dV = \int_{-1}^{1} |t|^{(N-1)(N+2)/2} \int_{\Omega_{R-1,S}} |P(1)|dV = \frac{4}{N(N+1)} \int_{\Omega_{R-1,S}} P(1)dV.$$

(Note, by assumption, that $P(1) \geqslant 0$, because all the (real) roots have absolute value $\leqslant 1$, and the leading coefficient is positive.) When $S = 0$, we see that

$$\int_{\Omega_{N,0}^P} dV = \frac{4}{N(N+1)} \int_{\Omega_{N-1,0}} P(1)dV$$

$$= \frac{1}{N(N+1)} \frac{1}{(N-1)!} \int_{[-1,1]^N} \prod(1-x_i) \prod|x_i - x_j|dx_1 \ldots dx_{N-1}.$$

The latter is another Selberg integral; in fact, by a special case of result of Aomoto [2],

$$\frac{4}{N+1} \int_{[-1,1]^{N-1}} \prod(1-x_i) \prod|x_i - x_j|dx_1 \ldots dx_{N-1} = \int_{[-1,1]^N} \prod|x_i - x_j|dx_1 \ldots dx_N,$$

and hence

$$\int_{\Omega_{N,0}^P} dV = \frac{1}{N!} \int_{[-1,1]^N} \prod|x_i - x_j|dx_1 \ldots dx_N = \int_{\Omega_{N,0}} dV.$$

Of course, this is as expected, because a Perron polynomial with real roots is simply a polynomial with real roots. Let $D_{R,S}^P = \int_{\Omega_{R,S}^P} dV$. Using the evaluation of our Selberg integral in Theorem 1.8, we can compute the volume of Ω_N^P. Namely, we have:

LEMMA 1.10. *There is an equality*

$$\mathrm{Vol}(\Omega_N^P) = \frac{4}{N(N+1)}C_{N-1}(1,1) = \begin{cases} D_N/N, & N \text{ odd}, \\ D_N/(N+1), & N \text{ even}. \end{cases}$$

Proof. The first equality follows from the computation above, the second is an elementary manipulation with products of factorials. □

We can do a similar analysis of expectation of $|a_N|^{\alpha-1}$ on Ω_N^P as we did with Ω_N. Namely

$$\int_{\Omega_N^P} |a_N|^{\alpha-1} dV = \int_{-1}^{1} \int_{\Omega_{N-1}} |t|^{(N-1)(N+2)/2} |t|^{(\alpha-1)N} |a_{N-1}|^{\alpha-1} P(1) \, dV \, dt$$

$$= \int_{-1}^{1} |t|^{(N-1)(N+2)/2+(\alpha-1)N} dt \int_{\Omega_{N-1}} |a_{N-1}|^{\alpha-1} P(1) \, dV$$

$$= \frac{4C_{N-1}(\alpha,1)}{N(N+2\alpha-1)}$$

and hence

$$E\big(\Omega_N^P, |a_N|^{\alpha-1}\big) = \begin{cases} \displaystyle\prod_{k=0}^{m} \frac{1+2k}{\alpha+2k}, & N = 2m+1 \text{ odd}, \\[2ex] \displaystyle\frac{N+1}{N+2\alpha-1} \prod_{k=0}^{m-1} \frac{1+2k}{\alpha+2k}, & N = 2m \text{ even}. \end{cases}$$

For example, $E(\Omega_{21}^P, |a_{21}|) = \frac{88179}{524288}$, as mentioned earlier.

REMARK 1.11. One could equally do this calculation insisting that Perron numbers be positive, one simply replaces the integral over $[-1,1]$ by an integral over $[0,1]$. This makes no difference to the proof of Theorem 1.2.

1.9. Selberg integrals on subspaces

Let us consider the complex Selberg integral

$$C_N^-(\alpha) = \int_{\Omega_{0,N}} |a_{2N}|^{\alpha-1}.$$

Recall that $(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$. We have:

THEOREM 1.12. *Let* $K_N = \prod_{k=1}^{N-1} \frac{k!^3 (k+1)! 2^{4k+1}}{(2k+1)!^2}$. *Then*

$$C_N^-(\alpha) = \int_{\Omega_{0,N}} |a_{2N}|^{\alpha-1} = K_N \cdot \binom{2N}{N} \cdot \frac{(4N+2\alpha)(N+\alpha+1/2)_N}{(N+\alpha)_{1+N}} \prod_{k=1}^{N} \frac{(\alpha+k)_{2N+1-2k}}{(\alpha-1/2+k)_{2N+2-2k}},$$

and $C_N^-(1) = \int_{\Omega_{0,N}} dV = D_{2N}^-$.

We prove this theorem in §3.1. It is possible to give explicit expressions for the integral $C_N^-(\alpha, T) = \int_{\Omega_{0,N}} |a_{2N}|^{\alpha-1} P(T)$ and $C_N^+(\alpha, T) = \int_{\Omega_{N,0}} |a_N|^{\alpha-1} P(T)$ as explicit polynomials whose coefficients are various products of Pochhammer symbols. However, since we have no use for them and the details are somewhat complicated, we omit them here.

1.10. *The expected number of real roots*

The expected number r_N of real roots of a random polynomial in Ω_N is given, for small N, as follows:

$$0, 1, \frac{2}{3}, \frac{17}{15}, \frac{32}{35}, \frac{43}{35}, \frac{1226}{1155}, \frac{1303}{1001}, \frac{10496}{9009}, \frac{208433}{153153}, \frac{402}{323}, \frac{1367}{969}, \ldots$$

There is a map:

$$\psi : \Omega_{N-1} \times [-1, 1] \to \Omega_N$$

given by sending $P(x)$ to $P(x)(x - T)$. The Jacobian of this map is equal to $|P(T)|$.

This map is not 1 to 1, rather, the image in $\Omega_{R,S}$ has multiplicity R. We exploit this fact as follows. Let $Z(P)$ denote the number of real roots of P. Then pulling back the measure under ψ^*, we deduce that

$$\int_{\Omega_N} Z(P) = \int_{-1}^{1} \int_{\Omega_{N-1}} |P(T)| dV.$$

For $|T| < 1$, the inner integral strictly differs from $C_{N-1}(1, T)$, so we cannot use our evaluation of $C_N(1, T)$ to compute this integral. On the other hand, if we compute a *signed* integral, then we are computing the probability that the number of real roots is odd. We thus deduce, as expected, that

$$\frac{1}{D_{N+1}} \int_{-1}^{1} C_N(1, T) = \begin{cases} 1, & N \text{ even,} \\ 0, & N \text{ odd.} \end{cases}$$

There is, however, the following conjectural formula for the integral of $|P(T)|$:

CONJECTURE 1. If $N = 2m$, then, for $T \in [-1, 1]$,

$$\frac{1}{D_N} \int_{\Omega_N} |P(T)| = \frac{1}{2^{2m} \binom{2m}{m}} \left(\sum_{k=0}^{m} \frac{2m - 2k + 1}{2m + 1} \binom{2m - 2k}{m - k} \binom{2k}{k} T^{2k} \right)$$

$$\times \left(\sum_{k=0}^{m} \binom{2m - 2k}{m - k} \binom{2k}{k} T^{2k} \right).$$

If $N = 2m + 1$, then, for $T \in [-1, 1]$,

$$\frac{1}{D_N} \int_{\Omega_N} |P(T)| = \frac{1}{2^{2m+2} \binom{2m}{m}} \left(\sum_{k=0}^{m} \frac{2m - 2k + 1}{2m + 1} \binom{2m - 2k}{m - k} \binom{2k}{k} T^{2k} \right)$$

$$\times \left(\sum_{k=0}^{m+1} \binom{2m + 2 - 2k}{m + 1 - k} \binom{2k}{k} T^{2k} \right).$$

Using this, we can say quite a bit about the explicit expectations r_N appearing above. There does not seem to be a closed form for r_N, but there is a nice relation between r_{2N+1} and r_{2N}, which comes down to an identity which may be proved using Zeilberger's algorithm.

LEMMA 1.13. *Assume Conjecture 1. There is an equality:*

$$(2N + 1 - r_{2N+1}) = \frac{4N + 3}{4N + 1}(2N - r_{2N}),$$

or

$$r_{2N+1} = \left(\frac{3 + 4N}{1 + 4N} \right) r_{2N} + \frac{1}{4N + 1}.$$

We now turn to asymptotic estimates of r_N. The exact arguments depend on the parity, but they are basically the same in either case (they are also equivalent by Lemma 1.13 above). Let us assume that $N - 1 = 2m$. Then

$$r_N = \frac{1}{D_N} \int_{\Omega_N} Z(P) = \frac{D_{N-1}}{D_N} \sum \sum \frac{1}{2^{2m} \binom{2m}{m}} \frac{2(2i+1)}{(2m+1)(4m-2i-2j+1)}$$
$$\times \binom{2i}{i} \binom{2j}{j} \binom{2m-2i}{m-i} \binom{2m-2j}{m-j},$$

which we can write as

$$\frac{(2m+1)!(2m)!}{2^{4m+1} m!^4} \sum \sum \frac{2(2i+1)}{(2m+1)(4m-2i-2j+1)} \frac{\binom{2i}{i} \binom{2j}{j} \binom{2m-2i}{m-i} \binom{2m-2j}{m-j}}{\binom{2m}{m}^2}.$$

Asymptotically,

$$\frac{(2m+1)!(2m)!}{2^{4m+1} m!^4} = \frac{1}{\pi} + \frac{m^{-1}}{4\pi} + \cdots.$$

Without this term, we can write the rest of the sum as:

$$s_N := \sum \sum \frac{2(2m-2i+1)}{2m+1} \frac{1}{2i+2j+1} \binom{2i}{i} \binom{2j}{j} 2^{-2i-2j} \prod_{k=m-i+1}^{m}$$
$$\times \left(1 + \frac{1}{2k-1}\right) \prod_{k=m-j+1}^{m} \left(1 + \frac{1}{2k-1}\right).$$

Let us estimate the contribution to this sum coming from terms where $i + j < A$ and $A = \epsilon \cdot m$. This will constitute the main term. For a lower bound, note that the final product is clearly at least one, and that i is at most A. Hence the sum is certainly bounded below by

$$\sum_{k<A} \sum_{i+j=k} 2 \left(1 - \frac{2A}{2m+1}\right) \frac{1}{2k+1} \binom{2i}{i} \binom{2j}{j} 2^{-2i-2j}.$$

Since

$$\sum_{i+j=k} \binom{2i}{i} \binom{2j}{j} 2^{-2i-2j} = 1,$$

the lower bound is at least

$$\sum_{k<A} 2 \left(1 - \frac{2A}{2m+1}\right) \frac{1}{2k+1} \geqslant (1 - \epsilon) \left(\log(m) + \log(\epsilon)\right) + O(1).$$

The $O(1)$ factor depends on ϵ, and should be thought of as an estimate as $m \to \infty$ with ϵ fixed. On the other hand, we have an upper bound for the sum given by noting that

$$\sum_{k=m-A+1}^{m} \log \left(1 + \frac{1}{2k-1}\right) \leqslant \sum_{k=m-A+1}^{m} \frac{1}{2k-1} = -\frac{1}{2} \log(1 - \epsilon) + o(1),$$

leading to an upper bound for the sum above of the form:

$$\sum_{k<A} \frac{2}{2k+1} \cdot \frac{1}{1-\epsilon} \leqslant \left(\frac{\log(m) + \log(\epsilon)}{1-\epsilon}\right) + O(1).$$

We now give an upper bound for the remaining sum. The initial factor is certainly at most 2, and $(2i + 2j + 1)^{-1} \leqslant (2A + 1)^{-1} \leqslant 1/(2A)$. Hence an upper bound is given by

$$\sum_{k \geqslant A} \sum_{i+j=k} \frac{1}{A} \binom{2i}{i} \binom{2j}{j} 2^{-2i-2j} \prod_{k=m-i+1}^{m} \left(1 + \frac{1}{2k-1}\right) \prod_{k=m-j+1}^{m} \left(1 + \frac{1}{2k-1}\right).$$

Note that we include terms in this sum with $k < A$ — this only increases the sum, but restores some symmetry. In particular, the sum is symmetric in $i \mapsto m - i$ and $j \mapsto m - j$, so after introducing a factor of 4, we may assume that i and j are both at most $m/2$. In this range, we have the upper estimate

$$\sum_{k=m-i+1}^{m} \log \left(1 + \frac{1}{2k-1}\right) \leqslant \sum_{k=m-\frac{m}{2}+1}^{m} \frac{1}{2k-1} \leqslant \frac{\log(2)}{2} + o(1).$$

Hence our remaining sum is bounded above by

$$\sum_{k} \sum_{i+j=k} \frac{4}{A} \binom{2i}{i} \binom{2j}{j} 2^{-2i-2j} \cdot 2 \leqslant \frac{8m}{A} = \frac{8}{\epsilon}.$$

Combining our two estimates, we find that, for ϵ fixed and $m \to \infty$,

$$1 - \epsilon + o(1) \leqslant \frac{s_N}{\log m} \leqslant \frac{1}{1-\epsilon} + o(1),$$

and thus $s_N \sim \log m \sim \log N$. Note that in order to obtain a better estimate (the second term, for example), we would have to be more careful about the dependence of the error terms on ϵ. The analysis for $N = 2m + 1$ is very similar. Hence, since $r_N \sim (1/\pi) \cdot s_N$, we deduce the following:

THEOREM 1.14.   *Assume Conjecture* 1. *If r_N denotes the expected number of real zeros of a polynomial in Ω_N, then, as $N \to \infty$,*

$$r_N \sim \frac{1}{\pi} \log N.$$

REMARK 1.15.   Note that, in the Kac model of random polynomials (where the coefficients are independent normal variables with mean 0), the expected number of zeros of a polynomial of degree N is $2/\pi \log N$, and the expected number in the interval $[-1, 1]$ is $1/\pi \log N$ [9]

Note that, for any $[a, b] \subset [-1, 1]$, the integral

$$\frac{1}{D_N} \int_a^b \int_{\Omega_N} |P(T)|$$

gives the expected number of zeros in the interval $[a, b]$. When $[a, b]$ does not contain either 1 or -1, the answer is particularly simple as $N \to \infty$.

THEOREM 1.16.   *Assume Conjecture* 1. *For a fixed interval $[a, b]$ with $-1 < a < b < 1$, the expected number of zeros in $[a, b]$ of a polynomial in Ω_N equals, as $N \to \infty$,*

$$\frac{1}{\pi} \int_a^b \frac{1}{1 - T^2} = \frac{1}{2\pi} \log \left| \frac{(1-a)(1+b)}{(1+a)(1-b)} \right|.$$

Proof. The argument is similar (but easier) to the asymptotic computation of r_N above. We consider the case $N - 1 = 2m$, the other case is similar. In some fixed interval $[a, b]$ not containing ± 1, the integrand converges uniformly to

$$\frac{1}{\pi}\left(\sum_{k=0}^{\infty}\frac{1}{4^k}\binom{2k}{k}T^{2k}\right)^2 = \frac{1}{\pi}\left(\frac{1}{\sqrt{1-T^2}}\right)^2,$$

from which the result follows. □

It is interesting to note that this is exactly the density function of real zeros for random power series [**11**]. This is also the limit density for Kac polynomials, which converge to random power series. This suggests that random polynomials in Ω_N might, in some sense to be made precise, approximate random power series for large N. This lends some credence to the speculations in 1.5.

2. *Counting polynomials*

Once we have formulas for volumes, the corresponding count of polynomials is fairly elementary.

LEMMA 2.1. *Let $\Omega \subset \mathbf{R}^N$ be a closed compact region such that $\partial\Omega$ is contained in a finite union of algebraic varieties of codimension 1. Let $\Omega(X) \subset \mathbf{R}^N$ denote image of Ω under the stretch $[X, X^2, \ldots, X^N]$. Then the number of lattice points in the interior $\Omega(X)$ is*

$$\mathrm{Vol}(\Omega)X^{N(N+1)/2}\left(1 + O\left(\frac{1}{X}\right)\right).$$

Proof. This essentially follows from 'Davenport's lemma' in [**3**] and [**4**]. With its conditions being met by our hypothesis, Davenport's result tells us that the main term is the volume of $\Omega(X)$, or $\mathrm{Vol}(\Omega)X^{N(N+1)/2}$. The error term should be $O(\max\{V_d(\Omega(X))\})$, where $\max\{V_d(\Omega(X))\}$ denotes the greatest d-dimensional volume of any of its projections onto a d-dimensional coordinate hyperplane, $\forall d \in \{1, 2, \ldots, n-1\}$. In our case, as X is large, this largest projection is clearly the one onto the last $N-1$ coordinates, which has volume proportional to $X^2 \cdot X^3 \cdots X^N$. Therefore the error term is $O(X^{N(N+1)/2-1})$. □

From this, it is easy to deduce:

LEMMA 2.2. *The number of irreducible polynomials of signature (R, S) all of whose roots have absolute value at most X is*

$$\mathrm{Vol}(\Omega_{R,S})X^{N(N+1)/2}\left(1 + O\left(\frac{1}{X}\right)\right).$$

The number of irreducible Perron polynomials of signature (R, S) all of whose roots have absolute value at most X is

$$\mathrm{Vol}(\Omega_{R,S}^P)X^{N(N+1)/2}\left(1 + O\left(\frac{1}{X}\right)\right).$$

Proof. The first claim without the assumption of irreducibility follows from the previous lemma. If a polynomial is reducible, however, then it factors as a product of two polynomials

each of which is monic (by Gauss) and has roots less than X by assumption. Hence the number of reducible factors is

$$\ll \sum_{\substack{A+B=N \\ A,B>0}} \mathrm{Vol}(\Omega_A)\mathrm{Vol}(\Omega_B)X^{A(A-1)/2+B(B-1)/2} = O\left(X^{N(N+1)/2-(N-1)}\right).$$

The argument for Perron polynomials is the same. $\square$

3. Evaluating integrals

We begin by evaluating the integral in Theorem 1.8. The integral $D_N = \int_{\Omega_N} dV$ was first computed by Fam in [7]. His method also allows one to easily compute $C_N(\alpha, T)$. Let $\Omega_N(a_N)$ denote the intersection of Ω_N with the hyperplane where the last coefficient is fixed. Then Fam shows that $\Omega_N(a_N)$ maps to Ω_{N-1} under a very explicit linear transformation. Explicitly,

$$\Omega_N(a_N) = T_{N-1}\Omega_{N-1},$$

where T_{N-1} is the following matrix:

$$T_{N-1} = I_{N-1} + a_N \widehat{I}_{N-1},$$

for the identity matrix I_{N-1} and the anti-diagonal matrix (1s on the anti-diagonal and 0s elsewhere) $\widehat{I}_{N-1}$. Explicitly, this takes the polynomial $Q(T) \in \Omega_{N-1}$ to the polynomial

$$P(T) = TQ(T) + a_N Q(1/T)T^{N-1} \in \Omega_N(a_N).$$

Recall that

$$C_N(\alpha, T) = \int_{\Omega_N} |a_N|^{\alpha-1}P(T)dV.$$

Then we have

$$C_N(\alpha, T) = \int_{-1}^{1}\int_{\Omega_N(a_N)} |a_N|^{\alpha-1}P(T)dV\,da_N.$$

Applying the change of coordinates above, we deduce that

$$C_N(\alpha, T) = \int_{-1}^{1}\int_{\Omega_{N-1}} |a_N|^{\alpha-1}(TQ(T) + a_N Q(1/T)T^{N-1})|\det(T_{N-1})|dV\,da_N.$$

By exchanging the order of integration and computing the determinant of T_{N-1}, it follows that

$$C_N(\alpha, T) = TA_N C_{N-1}(1, T) + T^{N-1}B_N C_{N-1}(1, 1/T),$$

where

$$A_N = \int_{-1}^{1}|t|^{\alpha-1}(1-t^2)^{(N-1)/2}dt,\ N \text{ odd},\ \int_{-1}^{1}|t|^{\alpha-1}(1-t^2)^{(N-2)/2})(1+t)dt,\ N \text{ even},$$

and

$$B_N = \int_{-1}^{1}|t|^{\alpha-1}t(1-t^2)^{(N-1)/2}dt,\ N \text{ odd},\ \int_{-1}^{1}|t|^{\alpha-1}t(1-t^2)^{(N-2)/2})(1+t)dt,\ N \text{ even}.$$

Both of these integrals are special cases of Euler's beta integral and may be evaluated explicitly. Theorem 1.8 follows easily by induction. The evaluation of the integral

$$S_N(\alpha, \beta) := \int_{\Omega_N} P(1)^{\alpha-1} P(-1)^{\beta-1}$$

is similar. The key point is that, under the substitution above with

$$P(T) = TQ(T) + a_N Q(1/T) T^{N-1},$$

we have

$$P(1) = Q(1) + a_N Q(1), \quad P(-1) = -Q(-1) + (-1)^{N-1} a_N Q(-1).$$

Recall that

$$\int_{-1}^{1} (1-t)^{a-1}(1+t)^{b-1} dt = 2^{a+b-1} \int_{0}^{1} (1-x)^{a-1} x^{b-1} dx = 2^{a+b-1} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Suppose that $N = 2m$ is even. Then

$$\int_{-1}^{1} (1+t)^{\alpha-1}(1+(-1)^N t)^{\beta-1} |\det(T_{N-1})| dt = \int_{-1}^{1} (1+t)^{\alpha-1}(1+t)^{\beta-1}(1+t)(1-t^2)^{m-1} dt$$

$$= \int_{-1}^{1} (1+t)^{\alpha+\beta+m-2}(1-t)^{m-1} dt = 2^{\alpha+\beta+2m-2} \frac{\Gamma(\alpha+\beta+m-1)\Gamma(m)}{\Gamma(\alpha+\beta+2m-1)}.$$

Suppose that $N = 2m+1$ is odd. Then

$$\int_{-1}^{1} (1+t)^{\alpha-1}(1+(-1)^N t)^{\beta-1} |\det(T_{N-1})| dt = \int_{-1}^{1} (1+t)^{\alpha-1}(1-t)^{\beta-1}(1-t^2)^{m} dt$$

$$= \int_{-1}^{1} (1+t)^{\alpha+m-1}(1-t)^{\beta+m-1} dt = 2^{\alpha+\beta+2m-1} \frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\alpha+\beta+2m)}.$$

By induction, we deduce the following:

THEOREM 3.1. *There is an equality:*

$$S_N(\alpha, \beta) = \prod_{2k \leqslant N} 2^{\alpha+\beta+2k-2} \frac{\Gamma(\alpha+\beta+k-1)\Gamma(k)}{\Gamma(\alpha+\beta+2k-1)} \prod_{2k+1 \leqslant N} 2^{\alpha+\beta+2k-1} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\alpha+\beta+2k)}$$

$$= \prod_{k=1}^{N} \frac{2^{\alpha+\beta+k-2}}{\Gamma(\alpha+\beta+k-1)} \prod_{2k \leqslant N} \Gamma(\alpha+\beta+k-1)\Gamma(k) \prod_{2k-1 \leqslant N} \Gamma(\alpha+k-1)\Gamma(\beta+k-1).$$

3.1. *The integral* $C_N^-(\alpha)$

LEMMA 3.2. *For non-negative integers* p_i *and* q_i, *we have*

$$\int_{B(1)^S} \prod_{i=1}^{S} |z_i|^{2(\alpha-1)} z_i^{p_i-1} \overline{z}_i^{q_i-1} \text{sign}(\text{im}(z_i))$$

$$= (\sqrt{-1})^S \cdot \begin{cases} \displaystyle\prod_{i=1}^{S} \frac{4}{2\alpha+p_i+q_i-2} \cdot \frac{1}{p_i-q_i}, & p_i - q_i \text{ all odd,} \\ \\ 0, & \text{else.} \end{cases}$$

Proof. The complex integral decomposes as a product over each $B(1)$. Changing to polar coordinates, we have

$$\int_{B(1)} z^{(\alpha-1)+(p-1)}\overline{z}^{(\alpha-1)+(q-1)}\text{sign}(\text{im}(z)) = \left(\int_0^\pi - \int_\pi^{2\pi}\right)\int_0^1 r^{2\alpha+p+q-3}e^{i(p-q)\theta}\,dr\,d\theta$$

$$= \frac{4}{2\alpha+p+q-2}\cdot\frac{\sqrt{-1}}{p-q},$$

if $p-q$ is odd and is trivial otherwise, from which the result follows. $\qquad\square$

For a complex number z, note that $|z-\overline{z}| = \sqrt{-1}\cdot\text{sign}(\text{im}(z))(\overline{z}-z)$. Thus the absolute value of the Vandermonde of signature (R,S) has the following expansion:

$$\prod_{i>j}|x_i - x_j|\prod(x_i - z_j)(x_i - \overline{z}_j)\prod_{i>j}|z_i - z_j|^2|z_i - \overline{z}_j|^2\prod|z_i - \overline{z_i}|$$

$$= (\sqrt{-1})^S\sum_{S_N}\text{sign}(\sigma)\prod_{i=1}^R x_i^{\sigma(i)-1}\prod_{j=1}^S z_j^{\sigma(R+2j-1)-1}\overline{z}_j^{\sigma(R+2j)-1}\text{sign}(\text{im}(z_j)).$$

Expanding out the integrand for $C_N^-(\alpha)$ term by term and integrating using Lemma 3.2 and then combining the two powers of $(\sqrt{-1})^N$, we find

$$C_N^-(\alpha) = \frac{1}{N!}\int_{\Omega_{0,N}}|a_N|^{\alpha-1}\,dV$$

$$= \frac{(-1)^N}{N!}\sum_{S_{2N}}\text{sign}(\sigma)\prod_{j=1}^N\frac{4}{(2\alpha+\sigma(2j-1)+\sigma(2j)-2)}\frac{1}{\sigma(2j-1)-\sigma(2j)}.$$

Here the sum is over permutations σ such that $\sigma(2j)$ and $\sigma(2j-1)$ are neither both odd nor even. Given any such σ, swapping the values of $\sigma(2j)$ and $\sigma(2j-1)$ changes the sign *both* $\text{sign}(\sigma)$ and $\sigma(2j-1)-\sigma(2j)$, and leaves everything else unchanged. Hence we may sum only over σ such that $\sigma(2j)$ is even, after including an extra factor of 2^N (the order of the stabilizer of the set of N unordered pairs). The set of permutations such that $\sigma(2j)$ is even is simply $S_N \times S_N$. The 'diagonal' copy of this group permutes the factors of the product. Hence we are reduced to the sum

$$\frac{2^N N!(-1)^N}{N!}\sum_{S_N\times 1}\text{sign}(\sigma)\prod_{j=1}^N\frac{4}{(2\alpha+\sigma(2j-1)+\sigma(2j)-2)}\frac{1}{\sigma(2j-1)-\sigma(2j)},$$

where now σ fixes the odd integers. Absorbing the 2^N and $(-1)^N$ factors into the product, we arrive at the sum

$$\sum_{S_N\times 1}\text{sign}(\sigma)\prod_{j=1}^N\frac{8}{(2\alpha+2j-1+\sigma(2j)-2)}\frac{1}{\sigma(2j)-2j+1}.$$

Yet this we may recognize as simply the Leibnitz expansion of the determinant of the following matrix:

$$M_N = \left[\frac{8}{(2\alpha+2i+2j-3)(2i-2j+1)}\right].$$

Hence $C_N^-(\alpha) = \det M_N$. This determinant is simply a Cauchy determinant. Let us write x_i for $2i = 2, \ldots, 2N$ and y_j for $2j - 1 = 1, \ldots, 2N - 1$, so then

$$M_N = \left[\frac{8}{(2\alpha + x_i + y_j - 2)(x_i - y_j)} \right] = \left[\frac{1}{X_i - Y_j} \right],$$

where

$$X_i = \frac{x_i^2 + 2\alpha x_i - 2x_i}{8}, \quad Y_j = \frac{y_j^2 + 2\alpha y_j - 2y_j}{8}.$$

Note that $X_i - X_j$ and $Y_i - Y_j$ factor into linear factors. By Cauchy's determinant formula, we may compute this determinant as a product of linear forms, and specializing as above we recover our formula for $C_N^-(\alpha)$ (after a certain amount of simplification with products of factorials).

3.2. The distribution of roots in Ω_N

In this section, we prove that the roots of almost all polynomials in Ω_N distribute nicely around the unit circle. We have already shown that the absolute values are close to one in Section 1.7, so to establish the radial symmetry we could apply the results of Erdös and Turán [6]. In fact, it is most convenient to prove both radial and absolute value distribution simultaneously using the nice formulation of [8]. The main point of that paper is to show that the quantity

$$F_N = \log \left(\sum_{i=0}^N |a_i| \right) - \frac{1}{2} \log |a_0| - \frac{1}{2} \log |a_N|$$

governs the behavior in both cases, and that it suffices to show that F_N is $o(N)$. In fact, we shall be able to obtain a much more precise estimate (logarithmic in N) which could be used to give quite refined qualitative bounds if so desired. Since $\log |a_0| = 0$ and $\log |a_N|$ was estimated in Section 1.7, the main task is to obtain bounds on the a_i. It is most natural to estimate integrals of the quantities $|a_i|^2$ over Ω_N, and this is what we do. We begin with a preliminary lemma.

LEMMA 3.3. *If N is even, then*

$$\frac{D_{N-1}}{D_N} \int_{-1}^1 a_N |\det(T_{N-1})| da_N = \frac{D_{N-1}}{D_N} \int_{-1}^1 a_N^2 |\det(T_{N-1})| da_N = \frac{1}{N+1}.$$

If N is odd, then

$$\frac{D_{N-1}}{D_N} \int_{-1}^1 a_N |\det(T_{N-1})| da_N = 0, \quad \frac{D_{N-1}}{D_N} \int_{-1}^1 a_N^2 |\det(T_{N-1})| da_N = \frac{1}{N+2}.$$

Let

$$A_N(i) = \frac{1}{D_N} \int_{\Omega_N} a_i dV.$$

Note that $A_N(i) = 0$ unless i is even. On the other hand, $A_N(i)$ is determined exactly by the coefficients of $C_N(1, T)$, namely, if $N = 2m + 1$, or $N = 2m$, then

$$A_N(2m - 2i) = \frac{2i + 1}{2m + 1} \frac{m!^2}{(2m)!} \binom{2i}{i} \binom{2m - 2i}{m - i}.$$

In either case, one has the (easy) inequality $|A_N(i)| \leqslant 1$. We now consider the integrals:

$$A_N(i, j) = \frac{1}{D_N} \int_{\Omega_N} a_i a_j dV.$$

If $i, j < N$, then:

$$D_N \cdot A_N(i,j) = \int_{\Omega_{N-1}} \int_{-1}^{1} (a_i + a_{N-i}a_N)(a_j + a_{N-j}a_N)|\det(T_{N-1})|da_N \, dV$$

$$= \int_{\Omega_{N-1}} a_i a_j dV \int_{-1}^{1} |\det(T_{N-1})|da_N$$

$$+ \int_{\Omega_{N-1}} (a_i a_{N-j} + a_j a_{N-i}) dV \int_{-1}^{1} a_N |\det(T_{N-1})| da_N$$

$$+ \int_{\Omega_{N-1}} a_{N-i} a_{N-j} dV \int_{-1}^{1} a_N^2 |\det(T_{N-1})| da_N.$$

If $i < N$ and $j = N$, then:

$$D_N \cdot A_N(i,N) = \int_{\Omega_{N-1}} \int_{-1}^{1} (a_i + a_{N-i}a_N)a_N |\det(T_{N-1})|da_N \, dV$$

$$= \int_{\Omega_{N-1}} a_i dV \int_{-1}^{1} a_N |\det(T_{N-1})| da_N$$

$$+ \int_{\Omega_{N-1}} a_{N-i} dV \int_{-1}^{1} a_N^2 |\det(T_{N-1})| da_N.$$

If $i = j = N$, then $A_N(i,j)$ is $1/(N+1)$ if N is even and $1/(N+2)$ if N is odd.

LEMMA 3.4.   *The inequality $|A_N(i,j)| \leqslant N^3$ holds for all i, j, N.*

Proof.   The result is certainly true for $N = 1$. We proceed by induction. Assume that neither i nor j is equal to N. By the recurrence relation above, the triangle inequality, and Lemma 3.3, we deduce that

$$|A_N(i,j)| \leqslant |A_{N-1}(i,j)| + \frac{1}{N+1} \left(|A_{N-1}(i, N-j)| + |A_{N-1}(j, N-i)| \right)$$

$$+ \frac{1}{N+1} |A_{N-1}(N-i, N-j)| \leqslant (N-1)^3 \left(1 + \frac{3}{N+1} \right) < N^3.$$

Something similar (but easier) occurs when either i or j is N (using the inequality $|A_N(i)| \leqslant 1$ noted above).   □

REMARK 3.5.   It seems from some light calculation that the inequality $|A_N(i,j)| \leqslant 2$ (or at least $O(1)$) may hold for all N, although we have not tried very hard to prove this, because the inequality above completely suffices for our purposes — indeed all that matters is that the bound is sub-exponential.

LEMMA 3.6.   *The following inequality holds for all N:*

$$\frac{1}{D_N} \int_{\Omega_N} \sum_{i=0}^{N} |a_i| \leqslant (N+1)^2 N^3.$$

Proof. By Cauchy–Schwartz,

$$\sum_{i=0}^{N} |a_i| \leqslant (N+1) \sqrt{\sum_{i=0}^{N} |a_i|^2} \leqslant (N+1) \sum_{i=0}^{N} |a_i|^2,$$

where we use the fact that $\sqrt{x} \leqslant x$ if $x \geqslant 1$ (note that $a_0 = 1$). It follows that

$$\frac{1}{D_N} \int_{\Omega_N} \sum_{i=0}^{N} |a_i| \leqslant (N+1) \sum_{i=0}^{N} |A_N(i,i)|,$$

and the result follows from Lemma 3.4. □

We now prove Theorem 1.2, which we restate now:

THEOREM 3.7. *As $N \to \infty$, the roots of a random polynomial in Ω_N or Ω_N^P are distributed uniformly about the unit circle.*

Proof. We first consider Ω_N. Following [**8**], consider the quantity

$$F_N = \log \left(\sum_{i=0}^{N} |a_i| \right) - \frac{1}{2} \log |a_0| - \frac{1}{2} \log |a_N|.$$

Note that $a_0 = 1$, so $\log |a_0| = 0$. This also implies that the first term in this sum is non-negative. On the other hand, certainly $|a_N| \leqslant 1$, so the last term is also non-negative, and $F_N \geqslant 0$ for every point of Ω_N. We have already computed that

$$-E(\Omega_N, \log |a_N|) = \tfrac{1}{2} \log N + O(1);$$

a similar result holds for Ω_N^P by the computation at the end of §1.8. Let $\Omega_N(100)$ denote the region where the first term in the above expression for F_N has absolute value at least $100 \log N$. (The constant 100 is somewhat arbitrary, it is relevant only that $100 > 5 + 1$.) By Lemma 3.6 we have

$$(N+1)^2 N^3 \geqslant \frac{1}{D_N} \int_{\Omega_N} \sum_{i=0}^{N} |a_i| \geqslant \frac{1}{D_N} \int_{\Omega_N(100)} N^{100} = \frac{\mathrm{Vol}(\Omega_N(100))}{\mathrm{Vol}(\Omega_N)} N^{100}.$$

It follows that the part of Ω_N where F_N is not between 0 and $100 \log |N|$ is a vanishingly small part of Ω_N for N large (by a large power of N). The same is true for Ω_N^P, because the volume of this latter space is (roughly) $1/N$ times the volume of Ω_N. The result then follows from [**8**, Theorem 1]. □

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