Research Proposal
Families of p-adic Modular Forms

1 Introduction

I work in algebraic number theory with an emphasis on the theory of modular forms, specifically, the interplay between modular forms and Galois representations. A recent theme in this area is the study of how such objects (both Galois representations and automorphic forms) deform in families.

My major proposed line of research involves the study of families of modular forms that are of cohomological type but are *not* associated to Shimura varieties. Of special interest is the case of modular forms over an imaginary quadratic field K. In this case, the associated symmetric space quotients are hyperbolic manifolds of real dimension 3, and thus, the study of such forms is not amenable to the usual techniques of algebraic geometry. A long-term goal of this project is to establish the modularity of elliptic curves over K. Although the Taylor–Wiles machinery is, in principle, applicable to this question, significant work is required to better understand the "Hecke" side of this theory, for example, the relationship between Hecke algebras of different level (for details, see § 2.1). The theory of modular forms over imaginary quadratic fields has several basic questions that remain open or partially unresolved, for which there is scant numerical data, in contrast to the situation for classical modular forms. There is great scope for student involvement on a range of hands-on projects; moreover, the relative simplicity required of these computations will accommodate students from a wide range of backgrounds. For details of the specific computations I have in mind, see §2.4. For possible student implementations, see §4.

The theory of modular forms over imaginary quadratic fields also has applications in low-dimensional topology. Let $\mathbf{A}^{\infty}$ denote the finite adeles, and $U \subseteq \mathrm{GL}_2(\mathbf{A}^{\infty}_K)$ a compact open subgroup. Let $\mathbb{H}^3$ denote the hyperbolic upper half space. Modular forms over K can be considered geometrically as sections of certain local systems on adelic quotients

$$X = \mathrm{GL}_2(K) \Big\backslash \big(\mathrm{GL}_2(\mathbf{A}^{\infty}_K)/U\big) \times \mathbb{H}^3.$$

Explicitly, such a space decomposes as a disjoint union of hyperbolic 3-manifolds $\mathbb{H}^3/\Gamma$, where Γ is commensurable with $\mathrm{GL}_2(\mathcal{O}_K)$, and thus, these manifolds are commensurable with Bianchi manifolds. Bianchi manifolds (and arithmetic manifolds in general) have been an active testing ground for several important questions in low-dimensional topology, such as the virtual positive Betti number conjecture [CL, Ro]. Recently, my work with Dunfield [CD] used the nonexistence of certain Galois representations to construct the first examples of rational homology spheres of arbitrarily large injectivity radius. Several topological questions have natural translations into intriguing problems about automorphic forms, suggesting a fertile area for interdisciplinary research. For details on some specific applications, please see §2.3.

My other proposed research area involves classical families of p-adic modular forms, in particular, the Coleman–Mazur eigencurve [CM]. Specific problems of interest include proving that the eigencurve is proper and studying the nonvanishing of p-adic zeta functions at arithmetic arguments. These ideas are detailed in §3.

2 Automorphic Forms of Cohomological Type

Automorphic forms come in a variety of flavors. A restrictive class of such representations are those which arise from algebraic geometry as cohomology classes of line bundles of some algebraic variety, such as a Shimura variety. These include many well-known examples, such as classical modular forms, as well as Hilbert modular forms and Siegel modular forms. The study of such objects often relies on this underlying algebraic structure; in particular, the corresponding Shimura varieties typically can be identified with moduli spaces for abelian varieties with PEL structure. A broader class of automorphic forms are those of "cohomological type." As the name suggests, they can be associated with certain cohomology groups, such as group cohomology or the cohomology of certain quotients of symmetric spaces. These spaces are not, in general, algebraic: they may be, for example, real manifolds of odd dimension. As in the case of automorphic forms arising from algebraic geometry, forms of cohomological type *do* conjecturally have associated Galois representations, although they do not exhaust the class of such forms (they exclude all Maass forms for $\mathrm{GL}_2/\mathbf{Q}$). This specific class of automorphic forms is not as well understood as those arising from algebraic geometry. On the other hand, there are a variety of tools of a quite different nature (homological methods, ideas from topology, and analytic methods such as base change) which are at play and have significant potential to replace algebraic arguments. In this discussion, we limit ourselves to modular forms over imaginary quadratic fields.

Let K be an imaginary quadratic field. One may define the notion of a *modular form* over K; such objects will be K-analogues of classical modular forms of weight $k \geq 2$. There are at least three equivalent ways to describe such forms. First, one may take real analytic functions on hyperbolic 3-space $\mathbb{H}^3$ satisfying certain transformational properties. (This is the most immediate analogue of classical modular forms, which are defined to be *complex* analytic functions on the upper half plane $\mathbb{H}^2$ satisfying prescribed functional equations.) Second, one may consider regular algebraic cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbf{A}_K)$. (This is the most natural definition from the perspective of base change.) Finally, one may consider cohomology classes in 3-manifolds $\mathbb{H}^3/\Gamma$ for congruence subgroups $\Gamma \subseteq \mathrm{GL}_2(\mathcal{O}_K)$. This last formulation is equivalent to considering the group cohomology of Γ for finite-dimensional irreducible representations of $\mathrm{GL}_2(\mathbf{C})$. In this last optic, one may consider the action of Hecke operators T_n for $n \in \mathcal{O}_K$, defined, as usual, in terms of double coset decompositions of Γ. These operators preserve the $\mathbf{Q}$-structure of cohomology and so have algebraic eigenvalues; moreover, their action can be simultaneously diagonalized, and the resulting eigenvectors are called modular eigenforms. The *weight* of a modular form is given by a finite-dimensional representation of $\mathrm{GL}_2(\mathbf{C})$. If $\mathbf{C}^2$ is the standard representation of $\mathrm{GL}_2(\mathbf{C})$, then let $\mathrm{Sym}^k(\mathbf{C}^2)$ denote its kth symmetric power. If V is any representation of $\mathrm{GL}_2(\mathbf{C})$, let $\overline{V}$ be the representation obtained from V by letting elements of $\mathrm{GL}_2(\mathbf{C})$ act via their complex conjugate. We say that a modular eigenform f has weight (k, k') if the system of Hecke eigenvalues associated to f occurs in

$$H^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbf{C}^2) \otimes \overline{\mathrm{Sym}^{k'-2}(\mathbf{C}^2)}).$$

A theorem of Harder [Ha] implies that there do not exist any cuspidal modular forms of weight (k, k') unless $k = k'$. The following theorem [HST, T_1, BHR] is the analogue of results of Shimura and Deligne for classical modular forms. Let G_K denote the absolute Galois group of $\overline{K}$ over K.

Theorem 1 (Taylor). *Let f be a normalized cuspidal modular eigenform over K of weight (k,k), with k even, whose central character χ is invariant under the action of* $\mathrm{Gal}(K/\mathbf{Q})$. *For all p, there exists a finite extension $E/\mathbf{Q}_p$ containing the Hecke eigenvalues of f and a continuous irreducible representation*

$$\rho_f : G_K \to \mathrm{GL}_2(E)$$

unramified outside a finite set of primes, such that for all but finitely many primes $v \in K$, the trace of $\rho_f(\mathrm{Frob}_v)$ equals the eigenvalue a_v of the Hecke operator T_v on f.

The cohomological theory admits an integral version that has no parallel in the language of automorphic representations. If $\Gamma \subseteq \mathrm{GL}_2(\mathcal{O})$ (with $\mathcal{O} = \mathcal{O}_K$), there is a natural representation of Γ on $\mathcal{O}^2$. Replacing the representations $\mathrm{Sym}^k(\mathbf{C}^2)$ by $\mathrm{Sym}^k(\mathcal{O}^2)$ yields cohomology groups which, tensored with $\mathbf{C}$, recover the usual space of modular forms, yet which can (and sometimes do) have torsion. Following Hida [H$_1$], one can package all these cohomology classes for fixed $\Gamma \subseteq \Gamma_1(\mathfrak{N})$ and varying weight together into a single module $\mathbf{H}$ over $\Lambda = \mathbf{Z}_p[[G]]$, where $G = \lim_{\leftarrow}(\mathcal{O}/\mathfrak{N}p^n\mathcal{O})^{\times}$, provided one restricts to *ordinary* forms. (In terms of the action of the Hecke operators, this corresponds to eigenforms whose T_v eigenvalue a_v is a p-adic unit for $v|p$; on the Galois side, this corresponds to the representations ρ_f [arising as in Theorem 1] being reducible after restriction to the decomposition groups at $v|p$.) By specializing $\mathbf{H}$ at certain primes $\mathfrak{p}$ of Λ (i.e., forming the modules $\mathbf{H} \otimes_{\Lambda} \Lambda/\mathfrak{p}$), one may recover all the classical spaces of ordinary modular forms for Γ. Since characteristic zero eigenclasses do not exist in integral weights (k,k') with $k \neq k'$, the module $\mathbf{H}$ is torsion over Λ. On the other hand, Hida has proved the following result [H$_1$]:

Theorem 2 (Hida). *There exists an exact sequence*

$$0 \to \Lambda^r \to \Lambda^r \to \mathbf{H} \to 0.$$

The Λ-module $\mathbf{H}$ is killed by a non-zero element in Λ and has homological dimension 1 over Λ.

It follows that the support of $\mathbf{H}$, if non-empty, has pure dimension one (considered inside the rigid analytic space of weights $\mathrm{Spf}(\Lambda)$, which is two-dimensional). In light of this, it is natural to speculate that the support of $\mathbf{H}$ consists exactly of parallel weights. We make this more precise. With respect to the action of $\mathrm{Gal}(K/\mathbf{Q})$, G decomposes as $G^+ \times G^- \times G^t$, where G^+ and G^- are free of rank one and G^t is finite. Let $\Lambda^{cycl} = \mathbf{Z}_p[[G^+]]$. There is a natural quotient map $\Lambda \to \Lambda^{cycl}$ making $\mathbf{H} \otimes \Lambda^{cycl}$ into a Λ^{cycl} module, which contains all the information from spaces of ordinary modular forms of parallel weight (k,k). Since these are the only integral weights in which the specialization of $\mathbf{H}$ can be non-torsion, it is tempting to conjecture that the support of $\mathbf{H}$ is exactly the diagonal, and that $\mathbf{H} \otimes \Lambda^{cycl}$ is free (or flat) over Λ^{cycl}. A consequence would be that every cuspidal eigenform lifts to a one-dimensional family of parallel weight cuspidal eigenforms. More explicitly, there would exist a collection of rigid analytic functions $a_n(T)$ (on some finite extension of Λ^{cycl}) such that for infinitely many specializations of T, the collection $\{a_n(T)\}$ is the set of Hecke eigenvalues of a classical cuspidal eigenform over K. Indeed, this is exactly what happens for classical modular forms (for $\mathrm{GL}_2/\mathbf{Q}$) and in many situations in which the underlying Shimura manifold is an algebraic variety. In [T$_2$], Taylor asked whether this was also the case for GL_2/K. We answer this question in the negative.

Theorem 3 (C). *The module* $\mathbf{H} \otimes \Lambda^{cycl}$ *is not flat over* Λ^{cycl} *in general and can have components of dimension* 0 *and* 1. *Not every ordinary cuspidal eigenform lifts to a one-dimensional* p*-adic family with infinitely many classical points.*

An implication of this theorem is that there exist one-dimensional families of Hecke eigenforms which contain *only finitely many classical points.* This theorem is in marked contrast to other situations considered previously, although it seems to be a consistent feature whenever one is outside the applicability of algebraic geometry (see for example [APS]).

Associated to $\mathbf{H}$, there exists, conjecturally, a family of ordinary Galois representations. Thus, one can ask for a Galois theoretic analogue of theorem 3. Given an ordinary Galois representation $\rho : G_K \to \mathrm{GL}_2(E)$, as in Taylor's theorem, does ρ admit ordinary deformations of parallel weight? Only when the answer to this question is positive can the corresponding family contain infinitely many representations arising from automorphic forms. Recently, Mazur and I have proved the following theorem (see [CaM]):

Theorem 4 (C, Mazur). *Let* $\rho : G_K \to \mathrm{GL}_2(\mathbf{C}) \simeq \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ *be an Artin representation unramified at* p *such that* $\rho(\mathrm{Frob}_v)$ *has distinct eigenvalues for* $v|p$. *Suppose that* ρ *admits infinitesimal ordinary deformations of minimal level and parallel weight. Assume the strong Leopoldt conjecture for the fixed field of* ρ. *Then either*

1. *Up to twist,* ρ *is the base change of an odd Artin representation over* $\mathbf{Q}$*, and the entire family is induced from a family of two-dimensional representations over* $\mathbf{Q}$*, or*

2. *The projective image of* ρ *is Dihedral; the corresponding quadratic extension* L/K *is a CM field (and thus bi-quadratic over* $\mathbf{Q}$*); and, the entire family is induced from a family of one-dimensional representations over* L.

The main theorem of [CaM] applies more generally to deformations of Artin representations over general number fields. In this generality, the only potentially classical deformations arise from inductions of one-dimensional representations over CM fields or via base change from totally real fields. The strong Leopoldt conjecture invoked in Theorem 4 is a genericity statement about the image of the unit group under the p-adic logarithm map inside the local units, which generalizes the classical Leopoldt conjecture.

2.1 Applications to Modularity

When proving modularity statements for ordinary representations, it is often no more difficult to work over the entire Hida family than in any particular classical weight. However, in the specific case of modular forms over some imaginary field K, there appear to be some obstructions to working in fixed weight that can be avoided by passing to ordinary families. To explain, we briefly recall some of the steps [W, TW] involved in proving an $R = \mathbf{T}$ theorem. We discuss what is required to adapt these methods to our particular context, and why working in Hida families may avoid some of the issues that arise in fixed weight. In this section, we suppose that $\overline{\rho} : G_K \to \mathrm{GL}_2(k)$ is a continuous irreducible representation, where k is a finite field, $\mathcal{O} = W(k)$, and E is the field of fractions of $\mathcal{O}$. We also suppose that $p = \pi \cdot \pi'$ splits in K.

1. We shall consider two Hecke rings, $\mathbf{T}$ and $\mathbf{T}_{(2,2)}$. Let $\mathbf{T}$ be the algebra generated by the endomorphisms T_n for every $n \in \mathcal{O}_K$ acting on $\mathbf{H}$, localized at the maximal ideal $\mathfrak{m}$ corresponding to $\overline{\rho}$. Let $\mathbf{T}_{(2,2)}$ be the algebra generated by the endomorphisms T_n acting on the space $H = H \otimes_\Lambda \Lambda/\mathfrak{p}_{2,2}$ of ordinary forms of weight $(2,2)$, localized at $\mathfrak{m}$. The ring $\mathbf{T}_{(2,2)}$ is a quotient of $\mathbf{T}$. The ring $\mathbf{T}$ conjecturally gives rise to a one-dimensional family of ordinary Galois representations, and the first step is to construct these representations. If R and $R_{(2,2)}$ denote the corresponding universal deformation rings, this step is necessary to prove the *existence* of maps $R \to \mathbf{T}$ and $R_{(2,2)} \to \mathbf{T}_{(2,2)}$.
2. *Level Lowering.* Given an irreducible representation $\overline{\rho}$ of Serre level $\mathfrak{N} = N(\overline{\rho})$, prove that $\overline{\rho}$ is modular of level $\mathfrak{N}$.
3. *Adding Auxiliary Primes.* In the classical context, Wiles shows that if Q is a set of auxiliary primes chosen in the usual way ($q \equiv 1 \mod p^n$, $\overline{\rho}(\mathrm{Frob}_q)$ has distinct eigenvalues), then $\mathbf{T}_Q$ is a free $\mathcal{O}[\Delta_Q]$-module, and hence $\mathbf{T}_\emptyset = \mathbf{T}_Q/\mathfrak{a}_Q$.

A central point to consider is that, even in parallel weight, the cohomology groups under consideration will often be torsion and nonzero. This is in contrast to situations involving Shimura varieties, in which the cohomology is either trivially torsion-free (i.e. the cohomology of surfaces) or in which torsion can be eliminated when $\overline{\rho}$ is not Eisenstein using Fontaine–Laffaille theory.

1. In the method of Taylor et al. [HST, T_1], it is essential to start with a characteristic zero eigenform, since the techniques involved are analytic (i.e. base change) rather than cohomological. Theoretically, there should be a torsion manifestation of base change, which we discuss in the context of p-adic endoscopy in §2.2. Another approach, due to Skinner and Urban, involves working with Eisenstein cohomology classes on $U(2,2)/K$ which correspond via base change to cusp forms for GL_2/K. Since self-adjoint classical cusp forms on $U(2,2)/K$ have associated Galois representations, one proceeds by constructing congruences between cusp forms and Eisenstein classes. Although at first this seems to reproduce only the result of Taylor (albeit without any Galois conditions on the central character), it has the potential to work in families and thus, to construct Galois representations over each connected component of $\mathrm{Spec}(\mathbf{T})$ that contains at *least one classical point.*
2. In this context, by "modular of level $\mathfrak{N}$," we mean the following: $\overline{\rho}$ occurs as an irreducible constituent in the cohomology of $H^1(\Gamma, \overline{V})$ for some $\mathbf{F}_p$-representation $\overline{V}$. Any characteristic zero formulation of Serre's conjecture will be false because $\mathbf{T}_{(2,2)}$ is, in general, not torsion free. Indeed, examples suggest that it is often the case that the Hecke ring $\mathbf{T}_{(2,2)}$ not only fails to be torsion free but can itself be torsion. On the other hand, one can try to prove results by *assuming* that $\overline{\rho}$ is modular of the level predicted by Serre's conjecture, and thus $\mathbf{T}_\emptyset \neq 0$.
3. Adding auxiliary primes does not seem to be a problem on the Galois side, for which many of the methods are relatively robust with respect to a change of base field (since they are mostly local calculations combined with Cebotarev density type arguments). On the other hand, arguments about Hecke rings often rely on geometric statements, such as certain multiplicity-one results, to ensure that various p-adic Tate modules (localized at the maximal ideal $\mathfrak{m}$ corresponding to $\overline{\rho}$) are free of rank two over the

Hecke ring. Even if analogous statements are true in this context, certain arguments break down when Hecke rings are not torsion-free. For example, the implication

$$\mathbf{T}_Q \text{ is a free } \mathcal{O}[\Delta_Q]\text{-module} \implies \mathbf{T}_\emptyset = \mathbf{T}_\mathbf{Q}/\mathfrak{a}_Q$$

usually proceeds by deducing that

$$\mathbf{T}_\emptyset \otimes_\mathcal{O} E = (\mathbf{T}_Q \otimes_\mathcal{O} E)/\mathfrak{a}_Q$$

and then noting that $\mathbf{T}_Q/\mathfrak{a}_Q$ is torsion free! If one works in fixed weight, then the rings $\mathbf{T}_{(2,2),Q}$ (which are not torsion free) cannot be free over $\mathcal{O}[\Delta_Q]$. Moreover, there is no reason to expect that $\mathbf{T}_{(2,2),Q} \otimes k$ is free over $k[\Delta_Q]$. If we work with the big Hecke algebra $\mathbf{T}$, then it is true that $\mathbf{T}$ *is* a torsion module over Λ. However, instead of working with $\Lambda = \mathbf{Z}_p[[G]]$ (where G is the inverse limit of the groups $(\mathcal{O}_K/\mathfrak{N}p^n\mathcal{O}_K)^\times$), the idea is to work instead over $\Lambda_\pi = \mathbf{Z}_p[[G_\pi]]$, where G_π is the inverse limit of the groups $(\mathcal{O}_K/\mathfrak{N}\pi^n\mathcal{O}_K)^\times$. There is no analogue of this step for $\mathbf{T}_{(2,2)}$. The first step in this program is to refine Hida's theorem to conclude that $\mathbf{T}$ is *flat* over Λ_π. The main ingredients of Hida's proof are Poincaré duality, the theory of regular sequences, and the ring theoretic properties of Λ. Note that Λ_π (which is essentially $\mathbf{Z}_p[[T]]$) is a very well-behaved ring. The second step is to use Hida's methods to show that $\mathbf{T}_Q$ is free over $\Lambda_\pi[\Delta_Q]$. Given such a statement, one may begin to apply the Taylor–Wiles machinery. A related approach is to use the intermediate Hecke ring $\mathbf{T}_{(2,?)}$, which controls ordinary forms of weight 2 at π and of unrestricted weight at π'. In this case, one expects that $\mathbf{T}_{(2,?)}$ is finite flat over $\mathcal{O}$ and that $\mathbf{T}_{(2,?),Q}$ is free over $\mathcal{O}[\Delta_Q]$. Several of the algebraic techniques used to study Hecke algebras for classical modular forms have been adapted to apply to cohomology or modular symbols and so, can be adapted to our context (see, for example, a recent preprint of Klosin [Kl]).

2.2 p-adic Endoscopy

It follows from Theorem 3 that if $\mathbf{T}$ denotes the Hecke algebra associated to $\mathbf{H}$, then there exist one-dimensional components of $\mathrm{Spec}(\mathbf{T})$ which contain only finitely many points corresponding to classical automorphic forms. This is counterintuitive to the usual philosophy that automorphic Galois representations are Zariski-dense in the space of all Galois representations. The following conjectural picture reconciles both views. Given $\mathrm{Spec}(\mathbf{T})$, one expects (see §2.1) the existence of a one-parameter family of two-dimensional ordinary representations of $\mathrm{Gal}(\overline{K}/K)$. These representations may be induced to yield families of four-dimensional representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. If one does this carefully (by first twisting with a suitable character of $\mathrm{Gal}(\overline{K}/K)$), the resulting representation can be massaged to land in $\mathrm{GSp}_4(E) \subset \mathrm{GL}_4(E)$. Assuming modularity conjectures for $\mathrm{GSp}_4/\mathbf{Q}$, the space of ordinary representations (after twisting) form a two-dimensional family on which classical points are dense. Thus, conjecturally, the one-dimensional family arising from $\mathrm{Spec}(\mathbf{T})$ should cut out a divisor $\mathscr{C}$ along this eigensurface. We deduce the following: even though this divisor may contain only finitely many arithmetic weights (and thus, only finitely many classical automorphic representations arising from base change from GL_2/K), each point on the divisor *does* lie in the closure of the classical points arising from cuspidal representations of $\mathrm{GSp}_4/\mathbf{Q}$. In other words, the associated Galois representations $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E)$ arising from $\mathrm{Spec}(\mathbf{T})$ can be thought of as a limit of automorphic representations $\varrho_n : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(E)$, each with open image in $\mathrm{GSp}_4(\mathscr{O}_E)$, yet congruent modulo increasing powers of p to an induced representation. This point of view also suggests a method for constructing p-adic L-functions along $\mathrm{Spec}(\mathbf{T})$, namely, by restricting p-adic L-functions for $\mathrm{GSp}_4/\mathbf{Q}$. In light of the fact that classical points may not be dense, the usual approach of constructing such functions (via special values of complex L-functions at classical points) does not work. A natural question is how to describe this divisor on the ordinary locus for GSp_4.

Conjecture 1 (C, Mazur)**.** *There exists a two-variable p-adic L-function on the GSp_4 eigencurve which p-adically continues the degree 5 L-function for GSp_4. The divisor $\mathscr{C}$ is equal to the polar divisor of $L_p(\pi \otimes \chi_K, 1)$, where χ_K is the quadratic character associated to K.*

This conjecture is inspired by Langlands' general philosophy of endoscopy, in which poles of complex L-functions correspond to copies of the trivial representation. One should note, however, that any general conjectures regarding *p-adic endoscopy* cannot be deduced from their complex analogues, since p-adic endoscopy would apply in situations without any classical points. The complex analogue of this conjecture is a theorem of Kudla, Rallis and Soudry [KRS], who prove that the classical degree-5 L-function $L(\pi \otimes \chi_K, 1)$ has a pole if and only if π comes from GL_2/K. There are various approaches to establishing this conjecture.

1. The definition of a p-adic L-function $L_p(\pi \otimes \chi_K, s)$ is not yet known, but should follow from a cross product of the work of Panchishkin [CoPa] (who defines such a p-adic L-function for a single $\mathrm{GSp}_4/\mathbf{Q}$ form) and the usual techniques of defining L-functions on eigencurves. Then, one may try to prove an integral version of [KRS], since theta lifts can in some settings be interpreted arithmetically.

2. An easier analogue, which is still interesting from the perspective of p-adic endoscopy, is the passage from GL_1/K to $\mathrm{GL}_2/\mathbf{Q}$; here one must identify forms with CM by some order in $\mathscr{O}_K$ in terms of the two-variable p-adic symmetric square L-function on the eigencurve. This seems like a useful exercise to do before the more complicated case; moreover, it has the advantage that all the objects have already been defined.

3. Another approach relates to the Skinner–Urban idea of working with $U(2,2)/K$ families; here one wants to identify some L value as the congruence modulus between Eisenstein cohomology and cusp forms.

2.3 Applications to Topology

Let M be a closed, connected, orientable 3-manifold. We say that M is a homology sphere if $H_1(M, \mathbf{Z}) = 1$, and a *rational* homology sphere if $H_1(M, \mathbf{Q}) = 1$. Poincaré was the first to observe that there exist nontrivial homology spheres. There are many known constructions of rational homology spheres, including Dehn surgery on knot compliments. However, one has the following conjecture of Thurston:

Conjecture 2 (Virtual b_1-conjecture)**.** *Let M be a closed, connected 3-manifold with infinite $\pi_1(M)$. Then there exists a finite cover $\widetilde{M} \to M$ with positive first Betti number; equivalently,* $\dim H_1(M, \mathbf{Q}) > 0$.

Since (by Poincaré duality) classes in $H_1(M, \mathbf{Q})$ give rise to incompressible surfaces in M, this conjecture implies Waldhausen's virtual Haken conjecture, one of the outstanding problems in 3-manifold topology. In light of the work of Perelman [P_1, P_2], the unknown (and generic!)

case of Conjecture 2 is when M is hyperbolic. A naïve approach to Conjecture 2 is to show that any hyperbolic manifold of sufficiently large volume has positive b_1. This fails, however, as there exist rational homology spheres of arbitrarily large volume. Cooper refined this question by asking whether there exist rational homology spheres of arbitrarily large injectivity radius (see Kirby's problem list [K], Problem 3.58). (Recall that M has injectivity radius r if, around each point x in M, there exists a ball $B_x(r)$ of radius r and an isometric embedding $B_x(r) \rightarrow M$.) By studying the possible Galois representations attached to a particular tower of arithmetic 3-manifolds, N. Dunfield and I [CD] were able to answer Cooper's question, namely:

Theorem 5. *Let $K = \mathbf{Q}(\sqrt{-2})$. Assume the generalized Riemann hypothesis (GRH) and the compatibility of local and Global Langlands conjecture for GL_2/K. Then there exists an explicit tower of rational homology spheres of arbitrarily large injectivity radius.*

After this result appeared as a preprint, Boston and Ellenberg [BE] studied our particular example and produced an unconditional proof using the theory of p-groups. The approach suggests other fruitful interactions between the study of automorphic forms for imaginary quadratic fields and questions in low-dimensional topology, especially those related to arithmetic 3-manifolds, a subject of much historic and current interest [MR]. One long-standing question about arithmetic 3-manifolds (originating with Bianchi [Bi]) is the following:

Question 1. *For which fundamental discriminants $-D$ with $K = \mathbf{Q}(\sqrt{-D})$ does the Bianchi manifold $M = \mathbb{H}^3/\mathrm{PSL}_2(\mathcal{O}_K)$ have cuspidal cohomology?*

Question 1 was resolved by Rohfs [Ro] and Zimmert [Z]. In particular, M has cuspidal cohomology (see [BS] for some references) exactly when

$$D \notin \{-1, -2, -3, -5, -6, -7, -11, -15, -19, -23, -31, -39, -47, -71\}.$$

This is proved (roughly) by considering a Lefschetz trace formula with respect to the action of complex conjugation on M. On the other hand, there should be a reasonably direct proof using base change; namely, such cohomology classes are related to regular algebraic automorphic representations for $\mathrm{GL}_2(\mathbf{A}_K)$ that have cyclotomic central character and are unramified principal series at each finite prime. Yet one may construct such representations as follows. Consider a classical modular eigenform f in $S_2(\Gamma_1(D), \epsilon)$. If the character of ϵ is divisible by every prime dividing D, then the associated abelian variety A_f acquires good reduction everywhere over some finite extension of $\mathbf{Q}$. Choosing ϵ appropriately, we may force this extension to be $\mathbf{Q}(\sqrt{-D})$. By local Langlands, if π_f denotes the corresponding automorphic representation, then the base change of π_f to K will have all the desired properties. Note that in contrast to the manifolds considered in Theorem 5, the Bianchi manifolds are not compact and thus, in general, have non-cuspidal cohomology. The analogue of Conjecture 2 for hyperbolic 3-manifolds with cusps was resolved by Cooper, Long and Reid [CLR].

Another question worth considering from an automorphic perspective is the following:

Question 2. *For which fundamental discriminants $-D$ with $K = \mathbf{Q}(\sqrt{-D})$ does the Bianchi manifold fiber over the circle? For which $-D$ is the Bianchi manifold virtually fibered?*

The existence of such a fibration is equivalent to a nowhere-vanishing one-form ω on M. This turns out to be equivalent to the existence of an automorphic form of weight $(2, 2)$ that is nowhere vanishing. The most obvious potential source of such forms arises via base change, which can be described analytically via theta lifts in terms of the associated classical modular forms.

2.4 Computational Aspects

In order to study modular forms over K explicitly, it is useful to perform computations.

Proposition 1. *For any K, $\Gamma \subseteq \mathrm{GL}_2(\mathcal{O}_K)$, and weight (k,k), the space of cusp forms of level Γ and weight (k,k) are effectively computable.*

All previous published computations [GHM, Cre] were restricted to weight $(2,2)$. Note that it is not clear *a priori* that there should exist *any* forms of high weight other than those arising from base change or from Weil restriction of Grossencharacters (and twists of these forms). Indeed, such forms appear to be scarce. For example, when $K = \mathbf{Q}(\sqrt{-2})$ and $(k,k) = (4,4)$, the smallest known form not arising from base change or Weil induction has level $\Gamma \subset \mathrm{GL}_2(\mathcal{O}_K)$ with $[\mathrm{GL}_2(\mathcal{O}_K) : \Gamma] = 168$. When $K = \mathbf{Q}(\sqrt{-2})$ and $\Gamma = \mathrm{GL}_2(\mathcal{O}_K)$, the only forms of weight (k,k) in the range of computation ($k \leq 96$) are all base change forms. Various natural questions present themselves:

1. Compute the homology groups for various levels Γ and quadratic imaginary fields K, *along with the action of the Hecke operators*. Characteristic zero classes conjecturally correspond to motives, but torsion classes are also of interest because of the conjectural association with Galois representations. Specific questions to consider are as follows:
   (a) Does there exist a cusp form of weight (k,k) when $K = \mathbf{Q}(\sqrt{-2})$ which does *not* arise from base change? What happens for other imaginary quadratic fields K?
   (b) Can one verify that $R = \mathbf{T}$ in examples when $\mathbf{T}$ is a purely torsion ring?
   (c) Can one numerically verify cases of Serre's conjecture (and related conjectures, such as level lowering)?
2. Base change forms. Using the triangulation of small Bianchi manifolds, compute analytically (using theta lifts) some base change forms from classical weight 2 forms. When is the corresponding one-form nowhere-vanishing?
3. Verify (in the spirit of Cremona, Stein and Taylor [Cre, St, T_1]) that elliptic curves of small conductor over K are modular.
4. For various primes $\mathfrak{p}$ in $\mathcal{O}_K$, when is the space of weight $(2,2)$ cusp forms of level $\Gamma_0(\mathfrak{p})$ nonzero? What is the density of such $\mathfrak{p}$?
5. Test whether there is a Jacquet-Langlands correspondence between the integral cohomology of Bianchi manifolds and the compact arithmetic 3-manifolds associated to the corresponding quaternion algebras. Note that the Jacquet–Langlands theorem for $\mathrm{GL}_2(\mathbf{A}_K)$ only implies the existence of a relationship between rational cohomology groups. Early indications suggest that an integral correspondence does indeed exist.
6. Compute any space of modular forms over K when $[K : \mathbf{Q}] \geq 3$ is not totally real.

Several of these computations have been performed over totally real fields, yielding results similar to those over $\mathbf{Q}$. Over imaginary quadratic fields, however, the numerology is quite different. For example, in (4) above, it is conjectured that this space of cusp forms *is* zero infinitely often. Yet it is unclear as to whether the density of such primes should be zero, one, or something in between.

3 Geometry of the Eigencurve

Let p be prime, and let N be some positive integer coprime to p. In [CM], Coleman and Mazur construct a rigid analytic space $\mathscr{C}$ parameterizing overconvergent p-adic eigenforms of tame level N (due to certain simplifying assumptions, their construction is for $N = 1$ and p odd, but other references, for example [Buz$_2$, Em], apply in general). If $\mathcal{W}_N$ denotes the rigid analytic space of weights, then there is a natural projection $\pi : \mathscr{C} \to \mathcal{W}_N$, which is finite flat "in the domain" (see [CM]). Coleman and Mazur posed the following question regarding the eigencurve:

> "Do there exist p-adic analytic families of overconvergent eigenforms of finite slope parameterized by a punctured disc, and converging, at the puncture, to an overconvergent eigencurve of infinite slope?"

One natural way of interpreting this question is to require that the map $\pi : \mathscr{C} \to \mathcal{W}_N$ satisfy the universal criterion for properness. By abuse of notation, if the answer to this question is yes, then we say that π is proper (which cannot literally be true, since π has infinite degree).

One application of properness is the following. Coleman proved [Co$_1$, Co$_2$] that for any overconvergent eigenform f of finite slope, there exists a family of overconvergent eigenforms over some neighborhood of weight space deforming f. If π is proper in the above sense, it follows that this family can be specialized *to any other weight in the same connected component of* $\mathcal{W}_N$. The ability to specialize to any weight (particularly arithmetic weights) is a useful technique in Iwasawa theory.

My work with Buzzard [BC$_2$] establishes the result in the case when $N = 1$ and $p = 2$. The main idea was to exploit the tension inherent in the following two facts:

1. Finite-slope, overconvergent eigenforms of classical weight extend far into the supersingular region (see [BT], [Buz$_1$]).

2. Infinite-slope eigenforms cannot extend far into the overconvergent region (see [BC$_2$]).

One reason that these ideas do not lead to a direct proof is that they only make sense for forms of arithmetic weight. Recall that overconvergent modular forms of weight zero are said to have radius of convergence r (with $r \in \mathbf{Q}$ and $r < p/(p+1)$) if they extend to some affinoid $X[\rho]$ for any $\rho < r$, where $X[\rho]$ is defined in terms of the valuation of some (equivalently, any) lift of the Hasse invariant (see, for example, [Ka] or [Buz$_1$]). For forms of integral weight, the radius of overconvergence is similarly defined as those sections of $\omega^{\otimes k}$ that extend over such $X[\rho]$. However, for generic weight κ, there is no well-defined notion of a sheaf $\omega^{\otimes \kappa}$, and one instead has to use an analytic definition using Eisenstein series. This leads to several ad hoc definitions of the rate of overconvergence for a form of weight κ. The problem is that such definitions are not sufficiently geometric to allow a proof of statements 1 and 2 in full generality. Indeed, the techniques of [BC$_2$] rely on specific and explicit analytic estimates (as in [BC$_1$]) that one cannot expect to write down in general.

One possible approach to properness is to use purely local properties of $\mathscr{C}$, in particular, Colmez's theory of trianguline representations and his local analogue of $\mathscr{C}$. Using (φ, Γ)-modules and Kedlaya's theory of slopes, one may interpret properness in a potentially approachable way. However, it is difficult to identify a correct formulation of triangular (φ, Γ)-modules (the (φ, Γ) analogue of trianguline) over bases such as a Tate algebra $\mathbf{Q}_p\langle X \rangle$, rather than $\mathbf{Q}_p$. The only progress thus far in this direction has been the work of Bellaïche and Chenevier [BCh], who define triangular (φ, Γ)-modules over Artinian algebras (see [BCh], §2.3). Inspired by their construction, I have defined the notion of a p-overconvergent eigenform of weight t for nilpotent t (for example, with $t^{M+1} = 0$) and, more generally, with weights in Artinian algebras. One defines a modular form of weight zero *with coefficients in* $\mathbf{C}_p[t]/t^{M+1}$ to be formal expressions

$$\sum_{i=0}^{M} t^i \cdot a_i,$$

where a_i are overconvergent of weight zero. Secondly, one defines the Eisenstein series E_t of weight t as follows:

$$E_t = 1 + \sum_{i=1}^{M} t^i \cdot \left. \frac{d^i E_\kappa}{d\kappa^i} \right|_{\kappa=0}.$$

Finally, one defines a modular form of weight t with coefficients in $\mathbf{C}_p[t]/t^{M+1}$ to be a formal q-expansion $\sum_{i=0}^{M} t^i \cdot F_i$ whenever

$$\frac{\sum_{i=0}^{M} t^i \cdot F_i}{VE_t} = \sum_{i=0}^{M} t^i \cdot b_i$$

is a modular form of weight zero with coefficients in $\mathbf{C}_p[t]/t^{M+1}$.

These definitions are seen to be consistent and extend to a definition of weight for any non-reduced point of $\mathcal{W}_N$. Using this idea, we may study Coleman and Mazur's question above as follows: Suppose there exists a family of finite-slope forms specializing to an infinite-slope form at some weight κ. By specializing at an infinitesimally nearby weight $\kappa + t$, one constructs a p-adic overconvergent eigenform of weight $\kappa + t$ and slope λ, for some $\lambda \neq 0$ which is nilpotent (it's easy to see that by taking M sufficiently large, one can force λ to be nonzero). The goal is then to prove that such an eigenform cannot exist as part of a family of finite-slope eigenforms. To demonstrate the plausibility of such a claim, we consider what happens if one has a weight zero overconvergent eigenform with coefficients in $\mathbf{C}_p[t]/t^{M+1}$ and slope t, which lifts to an eigenform in $\mathbf{C}_p[[t]]$. Let A be such a form. Then writing

$$A = a_0 + t \cdot a_1 + t^2 \cdot a_2 + \ldots,$$

it follows, by equating coefficients, that $Ua_0 = 0$, $Ua_1 = a_0$, etc. On the other hand, the operator U increases overconvergence, and so one may conclude that the radius of convergence of a_1 is less than that of a_0; the radius of convergence of a_2 is less than that of a_1, etc. In particular, one can produce a contradiction if A lifts to a slope t-eigenform with coefficients in $\mathbf{C}_p[t]/t^{M+1}$ for all M and simultaneously has some uniform lower bound on the radius of convergence of a_n for each n. Unlike in [BC$_2$], this approach is less reliant on explicit representations of the space of overconvergent functions. Using this approach, I have recently proved the following theorem:

Theorem 6 (C). *Let p be prime, and let N be a tame level coprime to p. Then the map $\pi : \mathscr{C} \to \mathcal{W}_N$ is proper in weight zero.*

Remarks: The only analytical estimates required in this theorem follow from calculations similar to those of Katz [Ka]. I believe the argument should extend to other arithmetic weights. The general case is more subtle but certainly approachable via these methods.

[redacted]

4 Broader Impact

This project has the potential to generate new collaborations between number theorists and low-dimensional topologists. Although topologists have used techniques from algebraic number theory for many years (see [MR] and the many references therein), this project neatly dovetails with open questions in both areas and so may naturally attract researchers from both fields.

Throughout my career, I have benefited enormously from the opportunity to participate in undergraduate research experiences, not only by learning mathematics but also by learning about the mathematical process. Not every area of mathematics is suitable for study outside a PhD student's specific research area; one cannot expect younger graduate students, let alone undergraduate students, to quickly grasp the vast amount of algebraic geometry needed for certain areas of number theory. I believe that this area provides a wonderful source for student projects because so little is known about modular forms over imaginary quadratic fields, and so one can much more quickly discover and understand the open questions. Another positive aspect is that the subject draws on many diverse branches of mathematics. I feel this is a strength when working with undergraduates or graduate students because it gives them a variety of perspectives and techniques from which to learn (or, for less experienced students, more choices to find their comfort zone). Here are some of the possible projects I have in mind:

1. Group Theory. Let $-D$ be a fundamental discriminant and let $K = \mathbf{Q}(\sqrt{-D})$. By constructing a presentation for $\mathrm{PSL}_2(\mathcal{O}_K)$, one may compute the abelianization of the group $\mathrm{PSL}_2(\mathcal{O}_K)$. More realistically, one can start with a fixed K and a known presentation of $\mathrm{PSL}_2(\mathcal{O}_K)$, and compute the abelianization of this group and several of its congruence subgroups. Many of these computations would be new and interesting, and help shed light on some of the questions in §2.4.

2. Homology. Modular symbols are an explicit way of understanding a triangulation of certain arithmetic manifolds. Using only linear algebra and some basic singular homology, one can compute the homology groups of Bianchi manifolds and their congruence subgroups (provided, as above, that one has an explicit triangulation of the original manifold, but these are easy to obtain for small D). Note that for this computation, one does not need to find triangulations of the congruence covers. Moreover, the answers obtained here should be identical to those obtained above; thus, two different groups of students could try to compute these answers separately and compare. More advanced students could try to compute the Hecke action on these groups.

3. Algebraic Number Theory. One can try to find small number fields over K with little ramification and small Galois group (say Dihedral, or perhaps A_4). Then, conjecturally, there should be associated Galois representations. This is yet a third way to compute the same groups as above, at least in part (the reductions modulo 2 and 3).

4. Analysis. Compute numerically some modular forms over K and get a feel for their analytic properties; then, compare with modular forms over $\mathbf{Q}$. This may require more sophistication than the above projects, but again one encounters the "same objects" computed above from yet a different perspective.

Although I am a new faculty member at Northwestern University as of September 2006, I am already active in graduate and undergraduate life, as I am supervising graduate reading courses and helping to organize the Northwestern Putnam team. I look forward to taking on several graduate students, as well as running summer programs for advanced undergraduates and graduate students to implement some of these projects. For those who become involved in the summer projects, there is great scope to move on to accessible PhD problems.

I have also written an expository article on modular forms (in the form of a book review), for the Bulletin of the American Mathematical Society [C].