Automorphisms of S_6 and invariants of 6 points in $\mathbb{P}^d$

(partially based on Howard-Millson-Snowden-Vakil)

§1. Outer automorphisms of S_n

If G is a group:

$$\underbrace{1 \to Z(G) \to G \xrightarrow{g \mapsto (x \mapsto gxg^{-1})} \mathrm{Aut}(G)}_{\text{Easy, including } G/Z(G) \triangleleft \mathrm{Aut}(G)} \to \underbrace{\mathrm{Out}(G)}_{\text{definition of Out}} \to 1$$

is exact.

$$\begin{aligned} \mathrm{Inn}(G) &= \mathrm{im}(G \to \mathrm{Aut}(G)) \\ &= \ker(\mathrm{Aut}(G) \to \mathrm{Out}(G)) \\ &\cong G/Z(G). \end{aligned}$$

Proposition:

$$\mathrm{Out}(S_n) \begin{cases} = 1, & \text{if } n \neq 6; \\ < \mathbb{Z}/2, & \text{if } n = 6. \end{cases}$$

Proof Step 1. If $\varphi \in \mathrm{Aut}(S_n)$ takes transpositions to transpositions then $\varphi \in \mathrm{Inn}(S_n)$:

- $\varphi(1\,i)$, $\varphi(1\,j)$ do not commute for $i \neq j$, so are of the form (ab), (ac).
- a is independent of i,j: $\langle (ab),(bc),(ca)\rangle \cong S_3$, while $\langle (1\,i),(1\,j),(1\,k)\rangle \cong S_4$.
- $\varphi(1\,i) =: (f(1), f(i))$ is well-defined.
- Since $\langle (1\,i)\rangle = S_n$, $f \mapsto \varphi$

$$G \to \mathrm{Aut}(G).$$

Step 2: If $c \subset S_n$ is a conjugacy class of involutions and $|c| = \binom{n}{2}$ then

- either $n=6$ and c is the class of $(12)(34)(56)$
- or c is the class of transpositions.

Step 2', for $n \geq 8$ (possible infinite): If c is a conjugacy class of involutions, then either

- C is the class of transpositions or
- C contains 2 elements whose product has order ≥ 4:
  - either $(12)(3)(4)(56)(7)\ldots, (1)(2)(34)(5)(67)\ldots$
  - or $(12)(34)(56)(78)\ldots, (23)(45)(67)(81)\ldots$ $\square$.

<u>Proposition</u>: Suppose $\varphi \in \mathrm{Aut}(S_6)$. Then TFAE:

(a) $\varphi \in \mathrm{Inn}(S_6)$.

(b) $\varphi^{-1}(\mathrm{Stab}(i)) = \mathrm{Stab}(f(i))$

(for some $i \iff$ for every i).

(c) $\varphi^{-1}(\mathrm{Stab}(i)) < S_6$ is <u>not</u> transitive

(in the obvious action on $\{1, \ldots, 6\}$).

In particular, given $S_5 \cong G < S_6$ with G transitive, $\varphi : S_6 \curvearrowright (S_6/G)$ is non-trivial in $\mathrm{Out}(S_6)$.

Proof $(a) \iff (b) \Rightarrow (c)$ is easy.

<u>Proof</u> (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) is easy.

For (c) $\Rightarrow$ (b), note that other non transitive subgroups have size $\leq \# S_2 \times S_4$
$= 2 \times 4! < 5!$

For the coset action, $\varphi^{-1}(\mathrm{Stab}(eG)) = G$. $\square$

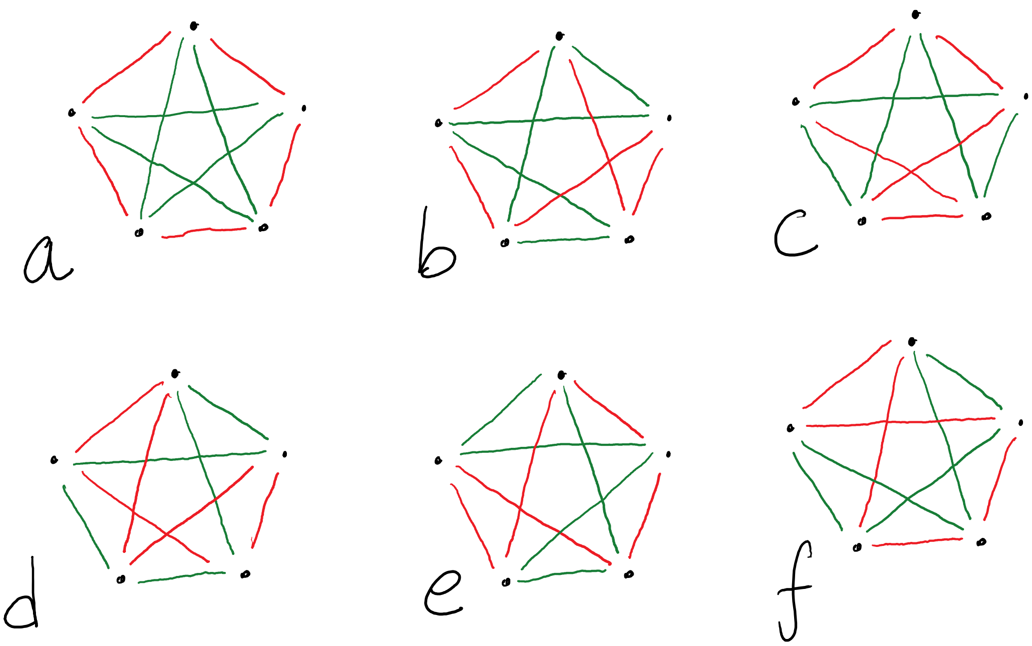
So we are looking for transitive (and faithful) S_5 actions on a 6 element set.

<u>Algebraic construction</u>: S_5 has 6 Sylow 5-subgroups (each generated by 4 of the 5-cycles). The action is by conjugation.

<u>Combinatorial/"mystical" interpretation</u>

Draw the 5-cycle $g \in S_5$ as a pentagon on given vertices, g^{-1} reverses orientation; g^2 and $g^{-2} = g^3$ form the complementary edges

$g = g$ form the complement of edges



The action of S_6 also has combinatorial interpretations:

- 20 triangles on 6 vertices. Color AB6 the same as the edge AB in the pentagon and CDE the opposite. These are all the 2-colorings such that...
- Glue the red triangles into half an icosahedron and symmetrize it. The green triangles glue to a 'complementary' icosahedron.

The 6-dimensional permutation represen tion

The 6-dimensional permutation representation associated to this action is the pullback of the defining permutation rep and decomposes as

$$\mathbb{1} \oplus V_5 \cong \mathbb{1} \oplus \varphi^* V_{\text{fund}}$$

$$V_5' := V_5 \otimes \mathbb{1}_{\text{sign}}$$

$$\operatorname{Res}^{S_6}_{S_5} V_5 = \cancel{\operatorname{Ind}^{S_5}_{\mathbb{Z}/5} \mathbb{1}}$$ is also irreducible.

§2. Moduli space of 6 points in $\mathbb{P}^d$

We are interested in the space

$$(\mathbb{P}^d)^6 /\!/ PGL_{d+1}$$

For $d > 3$, $PConf^6(\mathbb{P}^d)$ (the locus of 6 <u>distinct</u> $d \geq$ points) is a single dense orbit, so this space is trivial.

$d=1,2,3$ were all studied classically, and can be explicitly identified with projective varieties, as follows.

These identification maps are a generalization of the cross ratio!

$$(\mathbb{P}^1)^4 /\!/ PGL_2 \xrightarrow{\sim} \mathbb{P}^1.$$

<u>$d=1$</u>

$$(\mathbb{P}^1)^6 /\!/ PGL_2 \xrightarrow{\sim} \mathcal{S}_3 \subset \mathbb{P}^5$$

$$(p_1, \dots, p_6) \longmapsto [z_a : \dots : z_f]$$

given by $\displaystyle z_x = \sum_{i,j,k} \pm\, p_i p_j p_k$

$$H^0(\mathcal{S}_3, \mathcal{O}(1)) \cong V_5' \longleftrightarrow \left\{ \begin{array}{l} \text{depends on the color} \\ \text{of triangle } ijk \text{ given} \\ \text{by pentagon } x. \end{array} \right.$$

The image $\mathcal{S}$ the Segre threefold is

The image $\mathcal{S}_3$, the <u>Segre threefold</u> is cut out by $z_a + \cdots + z_f = z_a^3 + \cdots + z_f^3 = 0$.

<u>$d=3$ and duality with $d=1$:</u>

$$\begin{tikzcd}
(\mathbb{P}^3)^6/PGL_4 \arrow[rr, "\sim", "3"'] \arrow[d, dashed, leftrightarrow, "\wr", "2"'] & & I_4 \subset \mathbb{P}^5 \quad 1 \arrow[d, dashed, leftrightarrow, "\wr\ \text{dual hypersurfaces (in } \mathbb{P}^4)"] \\
(\mathbb{P}^1)^6/PGL_2 \arrow[rr, "\sim"] & & \mathcal{S}_3
\end{tikzcd}$$

1 I_4 is cut out by "Igusa quartic"

$$w_a + \cdots + w_f = \left(\sum w_x^2\right)^2 - 4\left(\sum w_x^4\right) = 0,$$

$$[w_a : \cdots : w_f] \in \mathbb{P}^5$$

2. 6 generic points in $\mathbb{P}^3$ are on a twisted cubic curve, i.e. a translate

2 . 6 generic points in $\mathbb{P}^3$ ...

unique rational normal curve, i.e. a translate of $\mathrm{im}\left(\mathbb{P}^1 \to \mathbb{P}^3;\ [s:t] \mapsto [s^3 : s^2t : st^2 : t^3]\right)$.

3
$$
\begin{aligned}
Z_{\mathbf{x}} = \sum_{(\sigma,\tau)\in S_5\times S_4} (\sigma,\tau)\circ\Big(&\frac{1}{2}w_2w_4w_6x_1x_2x_4y_1y_3y_5z_3z_5z_6 + w_1w_2w_4x_5x_6^2y_1y_2y_5z_3^2z_4\\
&-\frac{1}{2}w_2w_3^2x_5^2x_6y_2y_4^2z_1^2z_6 + 2w_2w_3w_4x_3x_5x_6y_4y_5y_6z_1^2z_2 - w_1w_2w_4x_3^2x_4y_5^2y_6z_1z_2z_6\\
&-\frac{2}{3}w_2w_5w_6x_3^2x_6y_1^2y_5z_2z_4^2 - \frac{1}{2}w_1w_2w_3x_1x_5x_6y_2y_3y_4z_4z_5z_6\\
&+\frac{1}{6}w_2w_3w_4x_1x_2x_5y_1y_4y_6z_3z_5z_6 + \frac{1}{4}w_1^2w_2x_2x_3^2y_5^2y_6z_4^2z_6\Big).
\end{aligned}
$$

The map $(\mathbb{P}^1)^6 \dashrightarrow I_4$ is easier to write down and provides $H^0(I_4; \mathcal{O}(1)) \cong V_5$.

$$W_x = \sum_{\substack{\sigma\in S_5\\ \{\alpha,\beta,\gamma\}=\{0,1,2\}}} (2 \text{ or } -1)\,(p_6 p_{\sigma(1)})^{\alpha}(p_{\sigma(2)}p_{\sigma(3)})^{\beta}(p_{\sigma(4)}p_{\sigma(5)})^{\gamma}$$

depends on (σ, x)

<u>$d=2$</u>:

$(\mathbb{P}^2)^6 /\!\!/ PGL_3 =: X$ is a double cover of $\mathbb{P}^4$ branched over I_4. The branch locus consists of 6 points on a conic, cut out by

$$V = \det(\nu_2(p_i))$$

Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$.

Then the double cover is given by

$$\underbrace{\left(\sum W_x^2\right)^2 - 4\left(\sum W_x^4\right)}_{I_4} + 324\, V^2 = 0.$$

where

$$W_x = \sum_{\sigma \in S_6} (2 \text{ or } -1)\,(x_{\sigma(1)} x_{\sigma(2)})(y_{\sigma(3)} y_{\sigma(4)})(z_{\sigma(5)} z_{\sigma(6)}).$$

depends on (σ, x) the same way as above.