# Negative Curves on Algebraic Surfaces

2020.5.9

# Outline

Introduction

- Bounded Negativity
  - Complex Surface with Surjective Endomorphisms
- Other Related Problems

### Introduction

## **Negative Curve**

By a *negative curve* we will always mean a reduced, irreducible curve with negative self-intersection.

### Conjecture

For each smooth complex projective surfaces X there exists a number  $b(X) \geq 0$  such that  $C^2 \geq -b(X)$  for every negative curve  $C \subset X$ .

# Answers to the Conjecture

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- If we consider the projective surface over an algebraic closed field with positive characteristic, then the answer is NO.
- Let C be a curve defined over an algebraically closed field k of characteristic p. Let X be the product surface  $C \times C$  and let  $\Delta$  be the diagonal. Let F be the Frobenius homomorphism. Then  $G = \operatorname{Id} \times F$  is a surjective homomorphism of X and  $\Delta, G(\Delta), G^2(\Delta), \cdots$  is a sequence of irreducible curves whose self-intersection tends to negative infinity.

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#### **Theorem**

Let X be a smooth projective complex surface admitting a surjective endomorphism that is not an isomorphism. Then X has bounded negativity.

- A surface *X* satisfying our hypothesis is one of the following types:
  - X is a toric surface.
  - X is a  $\mathbb{P}^1$ -bundle.
  - *X* is an abelian surface or a hyperelliptic surface.
  - X is an elliptic surface with Kodaira dimension  $\kappa(X)=1$  and topological Euler number e(X)=0

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  - X is a toric surface. In this case, effective cone is finitely generated (by Cox)
  - X is a  $\mathbb{P}^1$ -bundle. In this case, Picard number is 2. Hence effective cone is finitely generated.
  - X is an abelian surface or a hyperelliptic surface. For abelian surface  $K_X$  is trivial, for hyperelliptic surface  $-K_X.C \ge 0$ . Hence by genus formula,  $C^2 \ge -2$
  - X is an elliptic surface with Kodaira dimension  $\kappa(X)=1$  and topological Euler number e(X)=0

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#### Lemma

Let  $f: X \to Y$  be a finite morphism between two smooth surfaces X and Y. If Conjecture holds on X, then it holds on Y.

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### Proof.

Let C be an arbitrary curve on Y. Then

$$C^2 = \frac{1}{\deg(f)} (f^*C)^2 \ge \frac{-b(X)}{\deg(f)}$$

Hence the Conjecture holds with  $b(Y) = \frac{b(X)}{\deg(f)}$ .

#### Lemma

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#### Proof.

The group F acts on X naturally. If C is any smooth curve on X, If C is invariant under F, then it lies in the fiber of second projection, hence with zero self-intersection. If not, we can move C a little by F, hence the intersection is nonnegative.

#### Proof of the Theorem.

Let  $\pi: X \to B$  be an elliptic fibration, where B is a smooth curve. The condition e(X) = 0 implies the only singular fibers of X are possible multiple and the reduced fibers are always smooth elliptic curves.



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Let  $\pi: X \to B$  be an elliptic fibration, where B is a smooth curve. The condition e(X)=0 implies the only singular fibers of X are possible multiple and the reduced fibers are always smooth elliptic curves. Take a finite base change, we can resolve all the multiple fibers. Taking another finite base change if necessary, we obtain a fibration with smooth fibers and level n structure.

Such fibration is trivial since the moduli space of elliptic curves with level *n* structure is affine.



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#### **Theorem**

Let X be a smooth projective surface with  $\kappa(X) \geq 0$ . Then for every reduced, irreducible curve  $C \subset X$  of genus g(C), we have

$$C^2 \geq c_1^2(X) - 3c_2(X) + 2 - 2g(C)$$

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#### Theorem

For every integer m>0 there are smooth projective complex surfaces containing infinitely many smooth irreducible curves of self-intersection -m.

Let E be an elliptic curve without complex multiplication. Let A be the abelian surface  $E \times E$ ,  $F_1$  and  $F_2$  be the fiber of two projection and  $\Delta$  be the diagonal.

### **Fact**

Every elliptic curve on A that is not a translate of  $F_1$ ,  $F_2$  or  $\Delta$  has numerical equivalence class of the form

$$E_{c,d} := c(c+d)F_1 + d(c+d)F_2 - cd\Delta$$

where (c, d) = 1. And conversely, every such numerical class corresponds to an elliptic curve  $E_{c,d}$  on A.

Fix a positive integer t such that  $t^2 > m$ . For each  $E_n := E_{n,1}$  the number of t-division point is  $t^2$  and they are among the t-division point of A. We can find a subsequence of  $\{E_n\}$  such all its member has the same t-division points, say  $\{e_1, \dots, e_{t^2}\}$ .

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Take the blowup  $f: X \to A$  at  $\{e_1, \dots, e_m\}$  and let  $C_n$  be the strict transformation of  $E_n$ . Then

$$C_n^2 = E_n^2 - m = -m$$

For which d<0 and  $g\geq0$  is it possible to produce examples of surfaces X with infinitely many negative curves C of genus g such that  $C^2=d$ 

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#### **Theorem**

For every integer m > 1 and  $g \ge 0$  there are smooth projective complex surfaces containing infinitely many smooth irreducible curves of self-intersection -m and genus g.

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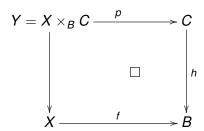
If *C* be any section of *f*, then  $C^2 = -\chi(\mathcal{O}_X)$ Pick any g > 0 and m > 2

### Fact

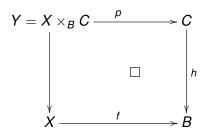
There is a smooth projective curve C of genus g and a finite morphism  $h: C \to B$  of degree m that is not ramified over points of B over which the fiber of f are singular.



We take



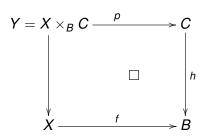
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Hence

$$\chi(\mathcal{O}_Y) = \frac{e(Y)}{12} = \frac{me(X)}{12} = m\chi(\mathcal{O}_X) = D^2$$

where D is any section of p. Then we only need to control  $\chi(\mathcal{O}_X) = -1$  which won't be hard.