Negative Curves on Algebraic Surfaces

2020.5.9

Outline

Introduction

Negative Curve

By a *negative curve* we will always mean a reduced, irreducible curve with negative self-intersection.

Conjecture

For each smooth complex projective surfaces X there exists a number $b(X) \geq 0$ such that $C^2 \geq -b(X)$ for every negative curve $C \subset X$.

Answers to the Conjecture

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- If we consider the projective surface over an algebraic closed field with positive characteristic, then the answer is NO.
- Let C be a curve defined over an algebraically closed field k of characteristic p. Let X be the product surface $C \times C$ and let Δ be the diagonal. Let F be the Frobenius homomorphism. Then $G = \mathrm{Id} \times F$ is a surjective homomorphism of X and $\Delta, G(\Delta), G^2(\Delta), \cdots$ is a sequence of irreducible curves whose self-intersection tends to negative infinity.

Outline

Theorem

Let X be a smooth projective complex surface admitting a surjective endomorphism that is not an isomorphism. Then X has bounded negativity.

Sketch of the Proof

- A surface X satisfying our hypothesis is one of the following types:
  - X is a toric surface.
  - X is a $\mathbb{P}^1$-bundle.
  - X is an abelian surface or a hyperelliptic surface.
  - X is an elliptic surface with Kodaira dimension $\kappa(X) = 1$ and topological Euler number $e(X) = 0$

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  - X is a $\mathbb{P}^1$-bundle. In this case, Picard number is 2. Hence effective cone is finitely generated.
  - X is an abelian surface or a hyperelliptic surface. For abelian surface K_X is trivial, for hyperelliptic surface $-K_X.C \geq 0$. Hence by genus formula, $C^2 \geq -2$
  - X is an elliptic surface with Kodaira dimension $\kappa(X) = 1$ and topological Euler number $e(X) = 0$

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Lemma

Let $f : X \to Y$ be a finite morphism between two smooth surfaces X and Y. If Conjecture holds on X, then it holds on Y.

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Proof.

Let C be an arbitrary curve on Y. Then

$$C^2 = \frac{1}{\deg(f)}(f^*C)^2 \geq \frac{-b(X)}{\deg(f)}$$

Hence the Conjecture holds with $b(Y) = \frac{b(X)}{\deg(f)}$. $\square$

Lemma

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Proof.

The group F acts on X naturally. If C is any smooth curve on X,If C is invariant under F, then it lies in the fiber of second projection, hence with zero self-intersection. If not, we can move C a little by F, hence the intersection is nonnegative. $\square$

Sketch of the Proof

Proof of the Theorem.

Let $\pi : X \to B$ be an elliptic fibration, where B is a smooth curve. The condition $e(X) = 0$ implies the only singular fibers of X are possible multiple and the reduced fibers are always smooth elliptic curves. $\square$

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Let $\pi : X \to B$ be an elliptic fibration, where B is a smooth curve. The condition $e(X) = 0$ implies the only singular fibers of X are possible multiple and the reduced fibers are always smooth elliptic curves.
Take a finite base change, we can resolve all the multiple fibers. Taking another finite base change if necessary, we obtain a fibration with smooth fibers and level n structure.
Such fibration is trivial since the moduli space of elliptic curves with level n structure is affine. $\square$

Problem

Is it possible to have a surface X with infinitely many negative curves C of bounded genus such that C^2 is not bounded from below?

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Theorem

Let X be a smooth projective surface with $\kappa(X) \geq 0$. Then for every reduced, irreducible curve $C \subset X$ of genus $g(C)$, we have

$$C^2 \geq c_1^2(X) - 3c_2(X) + 2 - 2g(C)$$

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Theorem

For every integer $m > 0$ there are smooth projective complex surfaces containing infinitely many smooth irreducible curves of self-intersection $-m$.

Sketch of the Proof

Let E be an elliptic curve without complex multiplication. Let A be the abelian surface $E \times E$, F_1 and F_2 be the fiber of two projection and Δ be the diagonal.

Fact

Every elliptic curve on A that is not a translate of F_1, F_2 or Δ has numerical equivalence class of the form

$$E_{c,d} := c(c+d)F_1 + d(c+d)F_2 - cd\Delta$$

where $(c, d) = 1$. And conversely, every such numerical class corresponds to an elliptic curve $E_{c,d}$ on A.

Sketch of the Proof

Fix a positive integer t such that $t^2 > m$. For each $E_n := E_{n,1}$ the number of t-division point is t^2 and they are among the t-division point of A. We can find a subsequence of $\{E_n\}$ such all its member has the same t-division points, say $\{e_1, \cdots, e_{t^2}\}$.

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Take the blowup $f : X \to A$ at $\{e_1, \cdots, e_m\}$ and let C_n be the strict transformation of E_n. Then

$$C_n^2 = E_n^2 - m = -m$$

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For which $d < 0$ and $g \geq 0$ is it possible to produce examples of surfaces X with infinitely many negative curves C of genus g such that $C^2 = d$

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Theorem

For every integer $m > 1$ and $g \geq 0$ there are smooth projective complex surfaces containing infinitely many smooth irreducible curves of self-intersection $-m$ and genus g.

Sketch of the Proof

Let $f : X \to B$ be a smooth complex projective minimal elliptic surface with section fibred over a smooth base curve B.

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Fact

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If C be any section of f, then $C^2 = -\chi(\mathcal{O}_X)$
Pick any $g \geq 0$ and $m \geq 2$

Fact

There is a smooth projective curve C of genus g and a finite morphism $h : C \to B$ of degree m that is not ramified over points of B over which the fiber of f are singular.

Sketch of the Proof

We take

$$\begin{tikzcd} Y = X \times_B C \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "h"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

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Hence

$$\chi(\mathcal{O}_Y) = \frac{e(Y)}{12} = \frac{me(X)}{12} = m\chi(\mathcal{O}_X) = D^2$$

where D is any section of p. Then we only need to control $\chi(\mathcal{O}_X) = -1$ which won't be hard.