Theta Functions, Old and New

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May 7, 2020

Line bundles on abelian varieties

Definition

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- Let T be an abelian variety
  $T = V/\Gamma$
  $V \cong \mathbb{C}^g, \quad \mathbb{Z}^{2g} \cong \Gamma$
- Want to understand the Picard group
  $$\begin{aligned}\mathrm{Pic}(T) &:= (\{\text{Complex line bundles on } T\}, \otimes) \\ &\cong H^1(T; \mathcal{O}_T^*)\end{aligned}$$

Exponential sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_T \xrightarrow{\exp} \mathcal{O}_T^* \longrightarrow 0$$

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$$\begin{tikzcd}
\mathrm{Pic}(T) \arrow[r, "c_1"] & H^2(T;\mathbb{Z}) \arrow[r] \arrow[d] & H^2(T;\mathcal{O}_T) \arrow[d, phantom, "\cong" rotate] \\
 & H^2(T;\mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \arrow[r] & H^{0,2}(T)
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$$\begin{aligned}
c_1(\mathrm{Pic}(T)) = NS(T) &:= H^2(T;\mathbb{Z}) \cap H^{1,1}(T) \\
&= \{E : \Lambda^2 V \to \mathbb{C} \mid \mathbb{Z}\text{-valued on } \Gamma,\ E(iz, iw) = E(z,w)\} \\
&= \{H : V^2 \to \mathbb{C} \text{ Hermitian} \mid E = \mathrm{Im(H)}\ \mathbb{Z}\text{-valued on } \Gamma\}
\end{aligned}$$

Exponential sheaf sequence, cont.

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$$\begin{tikzcd}
0 \arrow[r] & H^1(T;\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(T;\mathbb{R}) \arrow[r] \arrow[d, "\cong"] & H^1(T;S^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(T;\mathbb{Z}) \arrow[r] & H^{0,1}(T) \arrow[r] & \mathrm{Pic}^0(T) \arrow[r] & 0
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\end{array}$$

$$\therefore\ \mathrm{Pic}^0(T) \cong H^1(T;S^1) \cong \mathrm{Hom}(\Gamma, S^1)$$

Systems of multipliers

$$\begin{tikzcd} V \times \mathbb{C} \arrow[r] & \tilde{L} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d] \\ & V \arrow[r] & T \end{tikzcd}$$

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\begin{aligned}
L \cong & V \times \mathbb{C}/\Gamma \\
& \gamma \cdot (z, t) = (z + \gamma, e_\gamma(z)t) \\
& e_\gamma \in H^0(V; \mathcal{O}_V^*) \\
& e_{\gamma+\delta}(z) = e_\gamma(z + \delta)e_\delta(z)
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Remark. $\mathrm{Pic}^0(T) \cong H^1(\Gamma, H^0(V; \mathcal{O}_V^*))$

Systems of multipliers associated to Hermitian forms

$$\begin{aligned}\mathcal{P} := \Big\{(H,\alpha)\;\Big|\; & H \text{ Hermitian on } V,\\ & E = \operatorname{Im}(H)\ \mathbb{Z}\text{-valued on } \Gamma,\\ & \alpha : \Gamma \to S^1\\ & \alpha(\gamma+\delta) = \alpha(\gamma)\alpha(\delta)(-1)^{E(\gamma,\delta)}\Big\}\end{aligned}$$

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$$\begin{tikzcd}[column sep=small]
 & & \mathcal{P} \arrow[dr] & & \\
0 \arrow[r] & \operatorname{Hom}(\Gamma, S^1) \arrow[ur] \arrow[dr] & & NS(T) \arrow[r] & 0 \\
 & & \operatorname{Pic}(T) \arrow[ur] & &
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$$\begin{tikzcd} & & \mathcal{P} \arrow[d, "L", "\cong"'] \arrow[dr] & & \\ 0 \arrow[r] & \mathrm{Hom}(\Gamma, S^1) \arrow[ur] \arrow[dr] & \mathrm{Pic}(T) \arrow[r] & NS(T) \arrow[r] & 0 \end{tikzcd}$$

$L(H,\alpha)$ defined by multipliers $e_\gamma(z) = \alpha(\gamma)e^{\pi[H(\gamma,z)+\frac{1}{2}H(\gamma,\gamma)]}$

Theorem of the square

Lemma

For $a \in V$, let $t_a : T \to T$ be translation by a. Then $t_a^* L(H, \alpha) = L(H, \alpha')$, where $\alpha'(\gamma) = \alpha(\gamma) e^{2\pi i E(\gamma, a)}$.

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Theorem (Theorem of the square)

Let $L \in \mathrm{Pic}(T)$ *with Chern class* $E \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda^2 \Gamma, \mathbb{Z})$ *a unimodular form. Then the map*

$$\lambda_L : T \to \mathrm{Pic}^0(T)$$

$$\lambda_L(a) = t_a^* L \otimes L^{-1}$$

is an isomorphism.

Theta functions, polarizations

Definition

A *theta function* for the system of multipliers $\{e_\gamma\}$ defining is a holomorphic function θ on V satisfying

$$\theta(z+\gamma) = e_\gamma(z)\theta(z).$$

Equivalently, it is a section of the line bundle defined by those multipliers.

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Definition

A *polarization* of T is a *positive definite* Hermitian form on V such that $E = \mathrm{Im}(H)$ is $\mathbb{Z}$-valued on Γ. The polarization is *principal* if E is unimodular. We abbreviate "principally polarized abelian variety" by p.p.a.v.

Counting theta functions

Theorem

Let H be a principal polarization on T. Then for any α,

$$h^0(T; L(H,\alpha)^{\otimes d}) = d^g.$$

These sections are called "theta functions of degree d" for $L(H,\alpha)$.

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Proof idea.

- α is irrelevant by the theorem of the square.
- Choose a symplectic basis $\{\gamma_1, ..., \gamma_g, \delta_1, ..., \delta_g\}$ for E
- By choosing a particular α and modifying by a coboundary, find a system of multipliers for $L(H,\alpha)$ with $e_{\gamma_i}(z) = 1$.
- Fourier expand and use quasi-periodicity to relate the coefficients.

□

Explicit formulas

Definition

Let (T, H) be a p.p.a.v., and $\{\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g\}$ be a symplectic basis for E. Then the γ_i form a basis for V over $\mathbb{C}$, and the matrix τ expressing the δ_j in terms of the γ_i is the *period matrix*.

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Corollary

Having chosen a symplectic basis for a p.p.a.v., the unique theta function of degree 1 has formula

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i ({}^t m z + \frac{1}{2} {}^t m \tau m)}.$$

A basis for the theta functions of degree d is given by

$$\theta[\varepsilon](z) = \sum_{m \in \varepsilon} e^{2\pi i ({}^t m z + \frac{1}{2d} {}^t m \tau m)}, \quad \varepsilon \in \mathbb{Z}^g / d\mathbb{Z}^g$$

Consequences

- The existence of theta functions shows that any p.p.a.v. is a projective variety.

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- The existence of theta functions shows that any p.p.a.v. is a projective variety.
- A *Theta divisor* Θ is the vanishing locus of a theta function. By the theorem of the square, "the" theta divisor of a p.p.a.v. is well defined up to translation (but not up to linear equivalence).
- The explicit formula above is invariant under $i_T : z \mapsto -z$, so the corresponding theta divisor (denoted Θ_τ) is symmetric. Again from the theorem of the square we can deduce that $t_a^*\Theta_\tau$ is symmetric if and only if a is a 2-torsion point.

$|2\Theta|$

- It follows from the above that the linear system $|2\Theta|$ is the same for any of the 2^g symmetric divisors Θ, so this linear system is canonically associated to a p.p.a.v.

Theorem

For Θ symmetric and irreducible, the canonical map $\phi_{2\Theta} : T \to |2\Theta|^*$ factors through $T/i_T = K$ (the Kummer variety of T) and gives an embedding of K.

Jacobians

Let C be a genus g curve. Its *Jacobian*

$$JC \coloneqq H^{0,1}(C)/H^1(C;\mathbb{Z}) \cong \mathrm{Pic}^0(C)$$

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For each d, after fixing a degree d divisor D_d we can define the *Abel-Jacobi map*

$$\begin{aligned} C^{(d)} \coloneqq C^d/S_d &\xrightarrow{\alpha_d} JC \\ E &\longmapsto \mathcal{O}_C(E - D_d) \end{aligned}$$

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Theorem (Riemann)

The image of $\alpha_{g-1} : C^{(g-1)} \to JC$ is a theta divisor.

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Theorem (Riemann)

The image of $\alpha_{g-1} : C^{(g-1)} \to JC$ is a theta divisor.

Writing $L = \mathcal{O}_C(D_{g-1})$, this gives us another way of describing the theta divisor:

$$\Theta_L = \{M \in JC \mid H^0(M \otimes L) \neq 0\}$$

Higher rank

The role analogous to that of the Jacobian will be played by $\mathcal{M}(r)$, the moduli space of polystable rank r vector bundles E on C such that $\det(E) \coloneqq \Lambda^r E = \mathcal{O}_C$.

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Facts:

- $\dim \mathcal{M}(r) = (r^2 - 1)(g - 1)$
- $\mathcal{M}(r)$ is irreducible, normal, projective, with only mild singularities.
- $\mathcal{M}(r)$ is unirational. This is *very* far from being an abelian variety.

Generalized theta functions

Let J^d denote the space of line bundles of degree d. Fix some $L \in J^{g-1}$. In analogy with the theta divisor of the Jacobian, we define

$$\Delta_L = \{E \in \mathcal{M}(r) \mid H^0(E \otimes L) \neq 0\}.$$

Also, for each $E \in \mathcal{M}(r)$ we define the locus

$$\theta(E) = \{L \in J^{g-1} \mid H^0(E \otimes L) \neq 0\}$$

Theorem

$|\theta(E)|$ is either all of J^{g-1} or is a divisor in the linear system $|r\Theta|$.

Generalized theta functions, cont.

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It follows from the proof that θ defines a rational map $\mathcal{M}(r) \dashrightarrow |r\Theta|$. The corresponding linear system on $\mathcal{M}(r)$ is $|\Delta_L|$.

Generalized theta functions, cont.

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This map is reasonably well understood for $r = 2$. For example:

- For $g = 2$, $\theta : \mathcal{M}(2) \to |2\Theta|$ is an isomorphism.
- For $g \geq 3$, C not hyperelliptic, θ is an embedding.

Beyond that, very little is known!