## Theta Functions, Old and New

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### Line bundles on abelian varieties

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- Let T be an abelian variety  $T = V/\Gamma$  $V \cong \mathbb{C}^g$ ,  $\mathbb{Z}^{2g} \cong \Gamma$
- Want to understand the Picard group  $Pic(T) := (\{Complex line bundles on T\}, \otimes)$  $\cong H^1(T; \mathcal{O}_T^*)$

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$$c_1(\mathsf{Pic}(T)) = \mathit{NS}(T) \coloneqq H^2(T; \mathbb{Z}) \cap H^{1,1}(T)$$

$$= \{E : \Lambda^2 V \to \mathbb{C} \mid \mathbb{Z}\text{-valued on } \Gamma, \ E(iz, iw) = E(z, w)\}$$

$$= \{H : V^2 \to \mathbb{C} \ \text{Hermitian} \mid E = \mathsf{Im}(\mathsf{H}) \ \mathbb{Z}\text{-valued on } \Gamma\}$$

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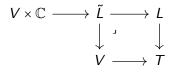
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$$0 \longrightarrow H^{1}(T; \mathbb{Z}) \longrightarrow H^{0,1}(T) \longrightarrow \operatorname{Pic}^{0}(T) \longrightarrow 0$$

$$\therefore \operatorname{Pic}^{0}(T) \cong H^{1}(T; S^{1}) \cong \operatorname{Hom}(\Gamma, S^{1})$$



$$L \cong V \times \mathbb{C}/\Gamma$$

$$\gamma \cdot (z, t) = (z + \gamma, e_{\gamma}(z)t)$$

$$e_{\gamma} \in H^{0}(V; \mathcal{O}_{V}^{*})$$

$$e_{\gamma+\delta}(z) = e_{\gamma}(z+\delta)e_{\delta}(z)$$

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Remark.  $\operatorname{Pic}^0(T) \cong H^1(\Gamma, H^0(V; \mathcal{O}_V^*))$ 

# Systems of multipliers associated to Hermitian forms

$$\mathcal{P} \coloneqq \{(H,\alpha) \mid H \text{ Hermitian on } V,$$
 
$$E = \operatorname{Im}(H) \ \mathbb{Z}\text{-valued on } \Gamma,$$
 
$$\alpha : \Gamma \to S^1$$
 
$$\alpha(\gamma + \delta) = \alpha(\gamma)\alpha(\delta)(-1)^{E(\gamma,\delta)}\}$$

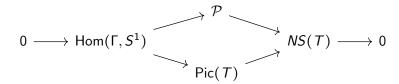
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$$0 \longrightarrow \operatorname{Hom}(\Gamma, S^1) \xrightarrow{\cong} \mathcal{L} \qquad NS(T) \longrightarrow 0$$

$$\operatorname{Pic}(T)$$

 $L(H,\alpha)$  defined by multipliers  $e_{\gamma}(z) = \alpha(\gamma)e^{\pi[H(\gamma,z) + \frac{1}{2}H(\gamma,\gamma)]}$ 

### Theorem of the square

#### Lemma

For  $a \in V$ , let  $t_a : T \to T$  be translation by a. Then  $t_a^* L(H, \alpha) = L(H, \alpha')$ , where  $\alpha'(\gamma) = \alpha(\gamma) e^{2\pi i E(\gamma, a)}$ .

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#### Theorem (Theorem of the square)

Let  $L \in Pic(T)$  with Chern class  $E \in Hom_{\mathbb{Z}}(\Lambda^2\Gamma, \mathbb{Z})$  a unimodular form. Then the map

$$\lambda_L: T \to \operatorname{Pic}^0(T)$$

$$\lambda_L(a) = t_a^* L \otimes L^{-1}$$

is an isomorphism.

# Theta functions, polarizations

#### Definition

A theta function for the system of multipliers  $\{e_\gamma\}$  defining is a holomorphic function  $\theta$  on V satisfying

$$\theta(z+\gamma)=e_{\gamma}(z)\theta(z).$$

Equivalently, it is a section of the line bundle defined by those multipliers.

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#### Definition

A polarization of T is a positive definite Hermitian form on V such that E = Im(H) is  $\mathbb{Z}$ -valued on  $\Gamma$ . The polarization is principal if E is unimodular. We abbreviate "principally polarized abelian variety" by p.p.a.v.

# Counting theta functions

#### Theorem

Let H be a principal polarization on T. Then for any  $\alpha$ ,

$$h^0(T;L(H,\alpha)^{\otimes d})=d^g.$$

These sections are called "theta functions of degree d" for  $L(H, \alpha)$ .

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#### Proof idea.

- ullet  $\alpha$  is irrelevant by the theorem of the square.
- $\bullet$  Choose a symplectic basis  $\{\gamma_1,...,\gamma_g,\delta_1,...,\delta_g\}$  for E
- By choosing a particular  $\alpha$  and modifying by a coboundary, find a system of multipliers for  $L(H, \alpha)$  with  $e_{\gamma_i}(z) = 1$ .
- Fourier expand and use quasi-periodicity to relate the coefficients.



## Explicit formulas

#### Definition

Let (T, H) be a p.p.a.v., and  $\{\gamma_1, ..., \gamma_g, \delta_1, ..., \delta_g\}$  be a symplectic basis for E. Then the  $\gamma_i$  form a basis for V over  $\mathbb{C}$ , and the matrix  $\tau$  expressing the  $\delta_j$  in terms of the  $\gamma_i$  is the *period matrix*.

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#### Corollary

Having chosen a symplectic basis for a p.p.a.v., the unique theta function of degree 1 has formula

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i \left( t_{mz} + \frac{1}{2} t_{m\tau m} \right)}.$$

A basis for the theta functions of degree d is given by

$$\theta[\varepsilon](z) = \sum_{m \in \varepsilon} e^{2\pi i (t_m z + \frac{1}{2d} t_m \tau m)}, \quad \varepsilon \in \mathbb{Z}^g / d\mathbb{Z}^g$$

# Consequences

• The existence of theta functions shows that any p.p.a.v. is a projective variety.

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   is the vanishing locus of a theta function.
   By the theorem of the square, "the" theta divisor of a p.p.a.v.
   is well defined up to translation (but not up to linear
   equivalence).
- The explicit formula above is invariant under i<sub>T</sub>: z → -z, so the corresponding theta divisor (denoted Θ<sub>τ</sub>) is symmetric.
   Again from the theorem of the square we can deduce that t<sub>a</sub>\*Θ<sub>τ</sub> is symmetric if and only if a is a 2-torsion point.

• It follows from the above that the linear system  $|2\Theta|$  is the same for any of the  $2^g$  symmetric divisors  $\Theta$ , so this linear system is canonically associated to a p.p.a.v.

#### Theorem

For  $\Theta$  symmetric and irreducible, the canonical map  $\phi_{2\Theta}: T \to |2\Theta|^*$  factors through  $T/i_T = K$  (the Kummer variety of T) and gives an embedding of K.

#### **Jacobians**

Let C be a genus g curve. Its Jacobian

$$JC \coloneqq H^{0,1}(C)/H^1(C; \mathbb{Z}) \cong \operatorname{Pic}^0(C)$$

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For each d, after fixing a degree d divisor  $D_d$  we can define the Abel-Jacobi map

$$C^{(d)} := C^d / S_d \xrightarrow{\alpha_d} JC$$

$$E \longmapsto \mathcal{O}_C(E - D_d)$$

### Riemann's Theorem

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#### Theorem (Riemann)

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#### Theorem (Riemann)

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Writing  $L = \mathcal{O}_C(D_{g-1})$ , this gives us another way of describing the theta divisor:

$$\Theta_L = \left\{ M \in JC \mid H^0(M \otimes L) \neq 0 \right\}$$

## Higher rank

The role analogous to that of the Jacobian will be played by  $\mathcal{M}(r)$ , the moduli space of polystable rank r vector bundles E on C such that  $\det(E) \coloneqq \Lambda^r E = \mathcal{O}_C$ .

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#### Facts:

- dim  $\mathcal{M}(r) = (r^2 1)(g 1)$
- $\mathcal{M}(r)$  is irreducible, normal, projective, with only mild singularities.
- $\mathcal{M}(r)$  is unirational. This is *very* far from being an abelian variety.

#### Generalized theta functions

Let  $J^d$  denote the space of line bundles of degree d. Fix some  $L \in J^{g-1}$ . In analogy with the theta divisor of the Jacobian, we define

$$\Delta_L = \{ E \in \mathcal{M}(r) \mid H^0(E \otimes L) \neq 0 \}.$$

Also, for each  $E \in \mathcal{M}(r)$  we define the locus

$$\theta(E) = \{ L \in J^{g-1} \mid H^0(E \otimes L) \neq 0 \}$$

#### Theorem

 $|\theta(E)|$  is either all of  $J^{g-1}$  or is a divisor in the linear system  $|r\Theta|$ .



### Generalized theta functions, cont.

#### **Theorem**

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It follows from the proof that  $\theta$  defines a rational map  $\mathcal{M}(r) oup |r\Theta|$ . The corresponding linear system on  $\mathcal{M}(r)$  is  $|\Delta_L|$ .

### Generalized theta functions, cont.

#### **Theorem**

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It follows from the proof that  $\theta$  defines a rational map  $\mathcal{M}(r) \rightarrow |r\Theta|$ . The corresponding linear system on  $\mathcal{M}(r)$  is  $|\Delta_L|$ .

This map is reasonably well understood for r = 2. For example:

- For g = 2,  $\theta : \mathcal{M}(2) \to |2\Theta|$  is an isomorphism.
- For  $g \ge 3$ , C not hyperelliptic,  $\theta$  is an embedding.

Beyond that, very little is known!