Polynomials and puzzle pieces

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Common roots of n polynomials in n variables

- Given a system of polynomial equations $f_1, \ldots f_n$ in n variables with coefficients in $\mathbb{C}$, how many common roots do they have in $\mathbb{P}^n$?
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Connections with polytopes

- Additional structure if take roots in $(\mathbb{C}^*)^n$
- **Newton polytope** of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$:
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Examples of Newton polytopes

- Single variable: $f(x) = a_r x^r + \ldots + a_s x^s$ with $r \leq s$, $N(f)$ line segment.
- How does this relate to roots of f?
- Earlier example: $N(g) = \text{conv}\{(0,0),(1,0),(1,1),(0,1)\}$ and $N(h) = \text{conv}\{(0,0),(2,1),(1,2)\}$.

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$= x^r(a_r + \ldots + a_s x^{s-r})$

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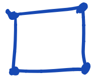

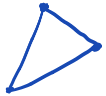

Why Newton polytopes?

- Hypersurface $Z_f := (f = 0) \subset (\mathbb{C}^*)^n$ vs. $\log : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$ sending $x \mapsto (\log|x_1|, \ldots, \log|x_n|)$
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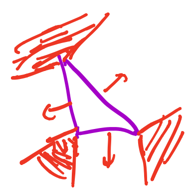

(translated) normal cones to N(f)

Roots and mixed volumes

- How to generalize single variable example from earlier? Why "volume"?
- Theorem (Bernstein): Given a "generic" choice of polynomials $g, h \in \mathbb{C}[x, y]$, the number of solutions to $g = h = 0$ in $(\mathbb{C}^*)^2$ is equal to the mixed volume $\mathcal{M}(N(g), N(h))$ of the Newton polytopes of g and h.

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Mixed volume example

- Given polygons P, Q (2D polytopes), the **mixed volume** is $\mathcal{M}(P, Q) = \text{Area}(P + Q) - \text{Area}(P) - \text{Area}(Q)$.
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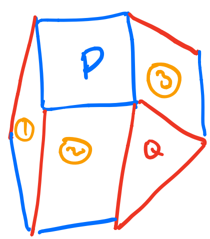

1, 2 3 -mixed

Bernstein's theorem sketch: Part 1

- Suppose g and h only have two distinct monomial terms (take one term as a constant)
- Use $SL_2(\mathbb{Z})$ to convert linear system to $x^a y^b = c, y^d = e$. Get ad roots, equal to $N(g) + N(h)$ (Newton polygons of sums of 2 monomials are lines)
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Bernstein's theorem sketch: Part 2

- Multiply terms by powers of t, get a **family** of systems of equations $g_t(x, y) = a_1 t^{v_1} + a_2 x t^{v_2} + a_3 xy t^{v_3} + a_4 y t^{v_4}$ and $h_t(x, y) = b_1 t^{w_1} + b_2 x^2 y t^{w_2} + b_3 x y^2 t^{w_3}$ (v_i, w_j "generic")
- In $(\mathbb{C}^*)^3$, solutions $(x(t), y(t))$ form a curve
- Permitted $v, w \in \mathbb{Q}$ for (Puiseux) power series expansions $x(t) = x_0 t^v +$ higher order terms and $y(t) = y_0 t^w +$ higher order terms ?

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- Let $P = N(g_t(x, y))$ and $Q = N(h_t(x, y)))$.
- A **lower facet** is a facet we can see from below (has an inward pointing normal vector $(u, v, 1) \in \mathbb{Q}^3$)
- Projection $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ sending $(x, y, t) \mapsto (x, y)$ induces bijection from lower hull to $N(g) + N(h)$

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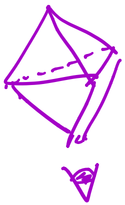

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- Projection also gives subdivision $\Delta = \{\pi(F) : F \text{ a lower facet of } P + Q\}$
- Given by projections F_1, F_2 of faces whose dimensions add to 2
- Mixed cells: Sums of projections of edges. Note: Sum of areas of mixed cells is $\mathcal{M}(N(g), N(h))$.

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- Mixed cell a sum of line segments/edges from each Newton polytope, come from binomial forms g', h'
- $g_t(x, y) = g'(x_0, y_0)t^a +$ higher order terms

  Lemma: Given lowest orders $(u, v) \in \mathbb{Q}^2$ for x and y, the corresponding $(x_0, y_0) \in (\mathbb{C}^*)^2$ are the nonzero roots of $g'(x_0, y_0) = h'(x_0, y_0)$.

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- Steps for computing zeros!
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