Polynomials and puzzle pieces

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These are notes following an exposition of Sturmfels ([3], Ch. 3 of [4]) on Bernstein's result ([1], [3], [4]) relating common roots of polynomials in $(\mathbb{C}^*)^n$ with mixed volumes of Newton polytopes.

1 COUNTING COMMON ROOTS OF POLYNOMIALS

Here is a basic question in algebraic geometry:

Question 1.1. *Given a system of polynomial equations $f_1, \ldots f_n$ in n variables with coefficients in $\mathbb{C}$, how many common roots do they have?*

Note that we set the number of equations equal to the number of variables since we want a finite number of solutions. Here is the usual (generic) answer:

Theorem 1.2. *(Bézout)*
For "generic" $f_1, \ldots, f_n$, there are $d_1 \cdots \cdots d_n$ common roots, where $d_i = \deg f_i$ counted with multiplicity.

As mentioned, this is good for "most" choices of $f_1, \ldots, f_n$ and these give distinct roots. However, we probably need a different method if we want to count the number of *distinct* roots. Even in the $n = 2$ case, it is unclear what is a practical way to do this. It is also hard to find the actual roots even if we were able to find the number of roots.

Here is a simple example where the expected number of common roots fails:

Example 1.3. (Sturmfels [3])
Let $n = 2$, $g = g(x, y) = a_1 + a_2x + a_3xy + a_4y$, and $h = h(x, y) = b_1 + b_2x^2y + b_3xy^3$. For a generic choice of coefficients a_i, b_j, it turns out that g and h only have 4 distinct common roots. This is smaller than what we expect from Bézout's theorem.

2 CONNECTIONS WITH POLYTOPES

In the case where we want to count roots in $(\mathbb{C}^*)^n$, there is some interesting combinatorial structure associated to these roots. From now on, we will focus on the $n = 2$ case. Note that there are analogous methods which work for larger n and even Laurent polynomials. The main new tool we will use are *Newton polytopes*, which are defined as follows:

Definition 2.1. A **Newton polytope** $N(f)$ of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is the convex hull in $\mathbb{R}^n$ of the lattice points recording the degree of each monomial.

Here are some examples of Newton polytopes:

Example 2.2.

- Suppose that $f(x) = a_r x^r + \ldots + a_s x^s$ with $r \leq s$. Then, the Newton polytope is the line segment in $\mathbb{R}$ connecting r and s. Note that there are generally $s - r$ nonzero roots of f, which is also the length of this line segment. What we will do with polytopes later will generalize this.

- In Example 1.3, we have that $N(g) = \text{conv}\{(0,0),(1,0),(1,1),(0,1)\}$ and $N(h) = \text{conv}\{(0,0),(2,1),(1,2)\}$.

Admittedly, it seems a bit unclear why we should think about the Newton polytope when we're thinking about common zeros of polynomials with ≥ 2 variables. We'll give a bit of motivation for this although it isn't necessarily directly related to the main result we will discuss.

The main general connection that the Newton polygon has to do with the structure of the hypersurface $Z_f := (f = 0) \subset (\mathbb{C}^*)^n$ has to do with the the map $\log : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$ sending $x = (x_1, \ldots, x_n) \mapsto (\log|x_1|, \ldots, \log|x_n|)$. Here are some facts related to this connection:

Fact 2.3. (p. 194 – 197 of [2])

- The "typical" picture of $\log Z_f$ (e.g. think about $f(x, y) = x + y + 1$) is bounded by walls made up of translated normal cones to the Newton polygon $N(f)$. These actually end up being "limiting directions" of $\log Z_f$ as we go further along (Proposition 1.9 of [2]).

- The vertices of the Newton polygon $N(f)$ correspond to connected components of $\mathbb{R}^k - \log Z_f$ containing an affine convex cone (Corollary 1.8 on p. 196 of [2]). This starts from considering convergence properties of the Laurent series of $\frac{1}{f}$.

3 COUNTING WITH VOLUMES

In the single variable example in Example 2.2, the number of roots is (usually) given by the length of the line segment which makes up the Newton polytope. The main result giving the relation we would like to discuss generalizes this to higher dimensions. Following Sturmfels' exposition [3], we will discuss the two-dimensional case via Example 1.3.

Theorem 3.1. *(Bernstein [3], [1])*
Given a "generic" choice of polynomials $g, h \in \mathbb{C}[x,y]$, the number of solutions to $g = h = 0$ in $(\mathbb{C}^*)^2$ is equal to the mixed volume $\mathcal{M}(N(g), N(h))$ of the Newton polytopes of g and h.

The term that stands out here is "mixed volume". Given polygons (2-dimensional polytopes) P and Q, the *mixed volume* is defined as $\mathcal{M}(P,Q) := \text{Area}(P+Q) - \text{Area}(P) - \text{Area}(Q)$, where $P+Q = \{p+q : p \in P, q \in Q\}$. Taking P and Q to be the Newton polygons from Example 1.3, we have a diagram for $P+Q$ with a subdivision induced by fixed translates of P and Q.

The mixed area is the area of the complement of the translates of P and Q. That is, they are the sums of the parallelograms with edges from P and Q). This subdivision will come up later.

Remark 3.2. In general, the mixed area is formally defined in a way similar to a determinant/volume except that the relations are symmetric instead of skew-symmetric. They also say something about how P and Q are oriented relative to each other since the mixed area changes after we keep the same shapes but just rotate one of them. There is also a more explicit way to define them involving volumes of linear combinations of polytopes, but we will not discuss them here.

We will now sketch the proof of Theorem 3.1.

Proof. (Proof sketch for Theorem 3.1)
Step 1: Case where g and h only have two distinct monomial terms.

Without loss of generality, we may assume that one of the terms is a constant for each of g and h since we're looking for solutions in $(\mathbb{C}^*)^2$ (factor out monomials). Write $g(x,y) = x^{a_1}y^{b_1} - c_1$ and $h(x,y) = x^{a_2}y^{b_2} - c_2$ for $a_i, b_j \in \mathbb{Z}$ and $c_k \in \mathbb{R}^*$. So, we would like to find common roots/solutions to $x^{a_1}y^{b_1} = c_1$ and $x^{a_2}y^{b_2} = c_2$. Since we're assuming $x, y \in (\mathbb{C}^*)^2$, we can take log on both sides to get a linear system of equations from $a_i \log x + b_i \log y = \log c_i$. Then, we can apply an element of $SL_2(\mathbb{Z})$ on the left to get an equivalent linear system associated to a matrix of the form $\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}$. This clearly has $r_1 r_2 = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$. This is the area of $N(g) + N(h)$ since the Newton polygons are the lines from the origin to each (a_i, b_i). Note that the general case actually eventually reduces to this.

Step 2: A family of systems of equations and power series

In order to look at Example 1.3, we will consider a family of systems of equations and consider our original system of equations to be a generic element of this family. More specifically, we will look at the toric deformations given by multiplying the terms by powers of t. So, we will look at $g_t(x,y) = a_1 t^{v_1} + a_2 x t^{v_2} + a_3 xy t^{v_3} + a_4 y t^{v_4}$ and $h_t(x,y) = b_1 t^{w_1} + b_2 x^2 y t^{w_2} + b_3 xy^2 t^{w_3}$ for some fixed "generic" choice of v_i and w_j. We will explain this in more detail once we look at how to solve this system of equations in terms of t.

Given a fixed value of t, common roots of g_t and h_t are of the form $(x(t), y(t))$. As a function of three variables (x, y, t), the roots form a curve in $(\mathbb{C}^*)^3$. In a neighborhood of the origin, we can write the algebraic functions $x(t)$ and $y(t)$ as (Puiseux) power series of the form

$$x(t) = x_0 t^v + \text{ higher order terms}$$

and

$$y(t) = y_0 t^w + \text{ higher order terms}$$

for some $v, w \in \mathbb{Q}$ and $x_0, y_0 \neq 0$.

Figuring out suitable $v, w \in \mathbb{Q}$ such that substituting $x(t)$ and $y(t)$ into g and h make the lowest order term equal to 0 gives the connection to Newton polytopes. It also explains the genericity condition on v_i and w_j. Substituting the expansions for $x(t)$ and $y(t)$ above into $g = g(x, y)$ and $h = h(x, y)$, we have

$$g_t(x(t), y(t)) = a_1 t^{v_1} + a_2 x_0 t^{v+v_2} + a_3 x_0 y_0 t^{v+w+v_3} + a_4 y_0 t^{w+v_4} + \text{ higher order terms}$$

and

$$h_t(x(t), y(t)) = b_1 t^{w_1} + b_2 x_0^2 y_0 t^{2v+w+w_2} + b_3 x_0 y_0^2 t^{v+2w+w_3} + \text{ higher order terms .}$$

Since $x_0, y_0 \neq 0$, we need the lowest order terms to appear at least twice in order for them to cancel out and make the coefficient of the lowest order term 0. The genericity condition on v_i and w_j states that it is attained exactly twice. The resulting inequalities involving linear forms is how this relates to polytopes.

Let's go back to polytopes and look at $P = N(g_t(x, y))$ and $Q = N(h_t(x, y))$. A **lower facet** is a facet (i.e. a 2-dimensional face) of $P + Q$ which has a vector $(u, v) \in \mathbb{Q}^2$ such that $(u, v, 1)$ is an inward-pointing normal vector to $P + Q$. In other words, it minimizes a linear form $ux_1 + vx_2 + x_3$ and is essentially a face of the polytope we can see from below. The *lower hull* is the union of all the lower facets of $P + Q$. This is everything we can see from below. The lower hull actually has a bijection with $N(g) + N(h)$ via the projection $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ sending $(x, y, t) \mapsto (x, y)$.

The projection $\Delta := \{\pi(F) : F \text{ a lower facet of } P + Q\}$ induces a subdivision of $N(g) + N(h)$. This is called a *mixed subdivision* of P and Q. Each cell of a mixed subdivision is of the form $F_1 + F_2$:

- $F_1 = \{(u_i, v_i)\}$ a point of of $N(g)$ (i.e. $x^{u_i} y^{v_i}$ a monomial of g), F_2 the projection of a facet of Q

- F_1 the projection of an edge of P, F_2 the projection of an edge of Q (**mixed cells**) Note that the sum of the areas of the mixed cells is mixed area $\mathscr{M}(N(g), N(h))$. This follows from subdividing the projection of $P + Q$ via $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ into the cell types described here.

- F_1 the projection of a facet of P, $F_2 = \{(u_i, v_i)\}$ a point of $N(h)$ (i.e. $x^{u_i} y^{v_i}$ a monomial of h)

Putting this together with the power series exponent considerations, we obtain one side of the following lemma.

Lemma 3.3. *([3]) A pair $(u, v) \in \mathbb{Q}^2$ is a pair of orders of the lowest terms in a series solution $(x(t), y(t))$ of the equations $g_t(x, y) = h_t(x, y) = 0$ if and only if $(u, v, 1)$ is the normal vector to a mixed lower facet of the polytope $P + Q$.*

We will not cover the other implication of this lemma (which involves a version of the implicit function theorem).

Step 3: Obtaining solutions
Finally, we relate the mixed cell decomposition above with finding actual solutions. Since each mixed cell is a sum of line segments/edges from each Newton polytope (for g and h), we have that the each edge is a Newton polytope of a binomial of two terms of g or h which we write as $N(g')$ and $N(h')$. Each mixed cell is associated to $g' = h' = 0$. Substituting the power series expansions of $x(t)$ and $y(t)$ from above we have

$$g_t(x, y) = g'(x_0, y_0)\, t^a + \text{ higher order terms}$$

and

$$h_t(x, y) = h'(x_0, y_0)\, t^b + \text{ higher order terms}$$

for some $a, b \in \mathbb{Q}$. Summarizing, we have

Lemma 3.4. *(Lemma 2.4 of [3]) Let (u, v) as in the Lemma 3.3. Then, the corresponding choices of $(x_0, y_0) \in (\mathbb{C}^*)^2$ are exactly the nonzero roots of $g'(x_0, y_0) = h'(x_0, y_0) = 0$.*

Step 1 implies that the number of nonzero roots in this lemma is $\text{Area}(N(g') + N(h'))$. Since these roots give the leading coefficients of the power series for $x(t)$ and $y(t)$ and every solution actually comes from a mixed cell, we can get all our solutions from the series. Counting over mixed cells, this gives us $\mathcal{M}(N(g), N(h))$.

□

It is interesting that this can give us an algorithm for obtaining roots in this case. Note that different mixed cells can give us different numbers of roots.

References

[1] D. N. Bernstein, The number of roots of a system of equations (Russian), Funk. Anal. Priloz. 9 (1975) 1 – 4, English translation in Functional Anal. Appl 9 (1975): 183 – 185.

[2] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, Discriminants, Resultants, and Multidimensional Determinants, Birkhäuser, Boston, MA, 1994.

[3] B. Sturmfels, Polynomial Equations and Convex Polytopes, The American Mathematical Monthly 105(10)(1998), 907 – 922.

[4] B. Sturmfels, Solving systems of polynomial equations, No. 97, American Mathematical Society, 2002.