Fermat Hypersurfaces

Weil considers : $a_0 x_0^{n_0} + \cdots + a_r x_r^{n_r}$ (attributes method to Hasse – Davenport , Hardy, Littlewood)

"Counts" the number of solutions in terms of Gauss Sums.

Main idea : use <u>characters</u> (both additive + multiplicative)

Exploit the symmetry of the hypersurface! Use linearity!

Gauss Sums : χ multiplicative character, ψ additive character

$$G_q(\chi, \psi) := \sum_{x \in \mathbb{F}_q} \chi(x)\, \psi(\mathrm{Tr}(x))$$

- $|G_q(\chi, \psi)| = \sqrt{q}$
- $G_{q^d}(\chi \circ \mathrm{Norm}, \psi) = G_q(\chi, \psi)^d \cdot (-1)^d$

Simplify : take all $n_i = n$. Then we can consider both the affine and projective variety.

Affine case : $\#\{ (x_i) \in \mathbb{F}_q^{r+1} \mid \sum_i a_i x_i^n = b \}$

Think of this as a function of b.

I.e. we have a measure / function, D, on $\mathbb{F}_q$, that assigns $b \mapsto \#\mathrm{Solns}\left(\sum_i a_i x_i^n = b \right)$

$$(x_i) \longmapsto \sum_i a_i x_i^n$$

$$f : \mathbb{A}^n(\mathbb{F}_q) \longrightarrow \mathbb{A}^1(\mathbb{F}_q)$$

D is the pushforward measure

$$D = f_* 1 \quad \text{i.e.} \quad D(x) = \sum_{y \in \mathbb{A}^n(\mathbb{F}_q)} 1$$

$f(y) = x$

D is generated in 3 steps:

$$\mathbb{A}^{r+1} \xrightarrow{f_1} \mathbb{A}^{r+1} \xrightarrow{f_2} \mathbb{A}^{r+1} \xrightarrow{f_3} \mathbb{A}^1$$

$$(x_i) \longmapsto (x_i^n) \qquad (x_i) \longmapsto (a_i x_i) \qquad (x_i) \longmapsto \sum_i x_i$$

Step 1+2 are multiplicative, step 3 additive.

Step 1 $r=1$ case:

$$\mathbb{F}_q \xrightarrow{x \to x^n} \mathbb{F}_q \xrightarrow{x \mapsto ax} \mathbb{F}_q$$

$$(f_{1*} 1)(x) = \begin{cases} 0 & \text{if } x \text{ is not an nth power} \\ 1 & \text{if } x = 0 \\ \underbrace{\# N_n(\mathbb{F}_q)}_{\gcd(q-1,n)} & \text{if } x \text{ is an nth power.} \end{cases}$$

Then we have:

$$f_{1*} 1 = \sum_{\substack{\chi : \mathbb{F}_q^* \to \mathbb{C}^* \\ \text{a character of} \\ \text{order } d}} \chi$$

where $\chi(0) = 0$ if χ non-trivial

$\chi_{triv} = 1$

if z not a n-th power, then $\sum \chi(z)$ is a sum over roots of unity $\Rightarrow$ vanishes

otherwise get $\gcd(q-1,n)$

Step 2: (still $r=1$ case)

Let χ be a character

$$f_{2*} \chi(t) = \chi(a^{-1} t) = \overline{\chi}(a) \chi(t).$$

Step 1+2: $r=1$ $\quad f_{2*} f_{1*} 1 = f_{2*} \sum_\chi \chi = \sum_\chi f_{2*} \chi = \sum_\chi \overline{\chi}(a) f_{2*} \chi$

Steps 1+2: r arbitrary

external product

$(\chi_0 \cdots \chi_r)(x_0, \ldots, x_r) = \prod_i \chi_i(x_i)$

r

$$f_{2*}f_{1*}\,1 = \prod_{i=0}^{r}\left(\sum_{\chi_i} \overline{\chi}_i(a_i)\,\chi_i\right) = \sum_{\chi_0,\dots,\chi_r} \overline{\chi}(a_0)\cdots\overline{\chi}(a_r)\cdot(\chi_0\cdots\chi_r).$$

Step 3: By linearity, suffices to consider $f_{3*}(\chi_0\cdot\cdots\cdot\chi_r)(b) = \sum_{b=z_0+\cdots+z_r} \prod_i \chi_i(z_i)$

But considered as functions this is just the convolution.

$$f_{3*}(\chi_0\cdot\cdots\cdot\chi_r) = \chi_0 * \cdots * \chi_r$$

Define $S(\chi_i) = (\chi_0 * \cdots * \chi_r)(0)$

$$\#\text{ Solutions} = \sum_{\substack{\chi_0,\dots,\chi_r \\ \mathbb{F}_q^* \to \mathbb{C}^* \\ \deg n}} \overline{\chi}_0(a_0)\cdots\overline{\chi}_r(a_r)\cdot S(\chi_i)$$

To compute convolution use fourier transform.

$$\chi_0 * \cdots * \chi_r = F^{-1}(F(\chi_0)\cdot\cdots\cdot F(\chi_r))$$

Fourier Transform:

Fix $\psi: \mathbb{F}_p \to \mathbb{C}^*$ additive character

$$F_\psi(f)(t) = \sum_{x\in\mathbb{F}_q} f(x)\,\overline{\psi}(Tr(tx))$$

$$F_\psi^{-1}(g)(t) = \frac{1}{q}\sum_{x\in\mathbb{F}_q} g(x)\,\psi(Tr(tx)) = \frac{1}{q}F_{\overline{\psi}}$$

$$F_\psi(\chi)(t) = \sum_{x\in\mathbb{F}_q} \chi(x)\,\overline{\psi}(Tr(tx)) = \sum_{y\in\mathbb{F}_q} \chi(y)\,\overline{\chi}(t)\,\overline{\psi}(t) = \overline{\chi}(t)\,G(\chi,\overline{\psi}) \quad \text{if } t\neq 0$$

$$\boxed{F_\psi(\chi)(t) = \begin{cases} \overline{\chi}(t)\,G(\chi,\overline{\psi}) & \text{if } \chi \text{ nontrivial} \\ q\,\delta_0 & \text{if } \chi \text{ trivial}\end{cases} \qquad \text{Here } \delta_0(t) := \begin{cases} 1 & \text{if } t=0 \\ 0 & \text{otherwise}\end{cases}}$$

Rmk: Applying $F_\psi^{-1} = \frac{1}{q}F_{\overline{\psi}}$ we see that $\chi(t) = \dfrac{G(\chi,\overline{\psi})\cdot G(\overline{\chi},\psi)}{q}\cdot\chi(t) \Rightarrow$ this product is 1.

Now $\prod_i F(\chi_i) = \begin{cases} q\,\delta_0 & \text{if all } \chi_i \text{ are trivial} \\ 0 & \text{if there is at least 1 trivial + nontrivial } \chi_i. \\ \prod_i G(\chi_i,\overline{\psi}) \cdot \prod_i \overline{\chi}_i(t) & \text{otherwise.} \end{cases}$

$$S(\chi_i) = \begin{cases} q^r & \text{if all } \chi_i \text{ are trivial} \\ 0 & \text{if some are trivial, some aren't.} \\ 0 & \text{if } \prod_i \chi_i \neq \text{trivial} \\ \dfrac{(q-1)\cdot\prod_i G(\chi_i,\psi)}{q} & \text{if } \prod_i \chi_i = \text{trivial.} \end{cases}$$

$\Rightarrow$ formula for

$\# \mathbb{V}(\sum_i a_i x_i^n) = 0$. To get the formula for the projective hypersurface, divide by $q-1$.

Cohomology of Fermat Hypersurface:

$X = \mathbb{V}(x_0^n + \cdots + x_r^n) \in \mathbb{P}^r$

For any smooth projective hypersurface

$$H^i(\mathbb{P}^n) \longrightarrow H^i(X) \quad \text{is an isomorphism for} \quad i \in [0, n-2] \cup [n, 2n-2]$$

(Weak Lefschetz + Poincaré Duality).

$$H^{prim}(X) := H^{n-1}(X) \big/ \operatorname{im} H^n(\mathbb{P}^n)$$

$\begin{cases} 0 & \text{if } n-1 \text{ odd} \\ \mathbb{Z} & \text{if } n \text{ even} \end{cases}$

Roots of unity, $(\mu_n)^{r+1}$ acts on X by $(\zeta_i)\cdot[x_0 : \cdots : x_r] = [\zeta_0 x_0 : \cdots : \zeta_r x_r]$

A character $(\mu_n)^{r+1} \longrightarrow \mathbb{C}^*$ $\quad \chi_0, \ldots, \chi_r \Rightarrow \left(H^{prim}(X) \otimes \mathbb{C}\right)_{\chi_0,\ldots,\chi_r}$ isotypic component

Thm:

$$\dim \left(H^{prim}(X)\otimes \mathbb{C}\right)_{\chi_0,\dots,\chi_r} = \begin{cases} 1 & \text{if } \prod_i \chi_i \text{ trivial} \\ 0 & \text{otherwise} \end{cases}$$

Thm: If $\mathbb{Z}[\zeta_n] \twoheadrightarrow \mathbb{F}_q$ is a finite field containing all roots of unity, then order-n characters of $\mathbb{F}_q^*$ correspond bijectively with characters of μ_n. And we have:

$$\mathrm{Tr}\left(\mathrm{Frob}, \left(H^{prim}_{et}(X_{\overline{\mathbb{F}}_q})\otimes\mathbb{C}\right)_{\chi_0,\dots,\chi_r}\right) = \frac{\prod_{i=0}^{r} G(\chi_i,\Psi)}{q}$$

Proof Sketch: Apply the twisted Grothendieck Lefschetz trace formula.

$$\mathbb{A}^{r+1} \xrightarrow{(x_i)\mapsto(x_i^n)} \mathbb{A}^{r+1} \quad \Rightarrow \quad \text{local system } \mathcal{L}_{\chi_0,\dots,\chi_r} \quad \text{corresponding to character.}$$

(ramified) cover

$$\mathbb{V}(x_0^n+\dots+x_r^n) \longrightarrow \mathbb{V}(x_0+\dots+x_r)$$

GLT $\Rightarrow$ $$S(\chi_0,\dots,\chi_r) = \mathrm{Tr}\left(\mathrm{Frob}, H^\bullet_c\left(\mathbb{V}\left(\sum_i x_i\right), \mathcal{L}_{\chi_0,\dots,\chi_r}\right)\right) \overset{GLT}{=} \sum_{x_0,\dots,x_r\in\mathbb{F}_q^{r+1}} \chi_0(x_0)\cdot\dots\cdot\chi_r(x_r)$$

Actually the second theorem + a chern class computation of $\dim H^{prim}(X)$ implies the first theorem. (All the eigenspaces of $(\mu_n)^r$ are nonzero, and $\#\{(y_i)\in(\mathbb{Z}/n-0)^{r+1} \mid \sum y_i = 0 \bmod n$

is the same as this dimension.

Example: $x^3+y^3+z^3=0$, Fermat Elliptic curve $E \subseteq \mathbb{P}^2$.

$$H^1(E) \cong H^1(E)_{1,1,1} \oplus H^1(E)_{2,2,2}$$

Say $p\equiv 1 \bmod 3$.

The two frobenius eigen-values are: $\dfrac{G(1,\Psi_p)^3}{p}$, $\dfrac{G(2,\Psi_p)^3}{p}$

$\dfrac{G(1,\Psi_p)^3}{p} \in \mathbb{Z}[\zeta_3]$ and has absolute value $\dfrac{p^{3/2}}{p} = p^{1/2}$.

$\Rightarrow$ $G(1, \bar{\psi}_p)$ is a generator of Q_1 for $p\mathbb{Z}[\zeta_3] = Q_1 \cdot Q_2$

narrows it down to one of 6 elts.